Lecture 12
Introduction to Wavelets
Last Time

- Started with STFT
- Heisenberg Boxes

- Continue and move to wavelets

- Ham exam -- see Piazza post
  - Please register at www.eastbayarc.org/form605.htm
Discrete STFT

\[ X[r, k] = \sum_{m=0}^{L-1} x[rR + m] \omega[m] e^{-j2\pi km/N} \]

\[ \Delta\omega = \frac{2\pi}{L} \]

\[ \Delta t = L \]

one STFT coefficient
Limitations of Discrete STFT

• Need overlapping $\Rightarrow$ Not orthogonal

• Computationally intensive $O(MN \log N)$

• Same size Heisenberg boxes
From STFT to Wavelets

• Basic Idea:
  – low-freq changes slowly - fast tracking unimportant
  – Fast tracking of high-freq is important in many apps.
  – Must adapt Heisenberg box to frequency

• Back to continuous time for a bit.....
From STFT to Wavelets

- Continuous time

\[ S f(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t - u)e^{-j\Omega t} \, dt \]

\[ W f(u, s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*\left(\frac{t-u}{s}\right) \, dt \]

*Morlet - Grossmann*
From STFT to Wavelets

\[ Wf(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^* \left( \frac{t - u}{s} \right) dt \]

- The function \( \Psi \) is called a mother wavelet
  - Must satisfy:
    \[
    \int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \quad \Rightarrow \text{unit norm}
    \]
    \[
    \int_{-\infty}^{\infty} \Psi(t) dt = 0 \quad \Rightarrow \text{Band-Pass}
    \]
STFT and Wavelets “Atoms”

**STFT Atoms**
(with hamming window)

\[ w(t - u)e^{j\Omega t} \]

\[ \Omega_{hi} \]

\[ \Omega_{lo} \]

**Wavelet Atoms**

\[ \frac{1}{\sqrt{s}} \Psi\left(\frac{t - u}{s}\right) \]

\[ s = 1 \]

\[ s = 3 \]

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Examples of Wavelets

- **Mexican Hat**
  \[
  \Psi(t) = (1 - t^2)e^{-t^2/2}
  \]

- **Haar**
  \[
  \Psi(t) = \begin{cases} 
  -1 & 0 \leq t < \frac{1}{2} \\
  1 & \frac{1}{2} \leq t < 1 \\
  0 & \text{otherwise}
  \end{cases}
  \]
Example: Wavelet of Chirp
Wavelets VS STFT
Example 2: “Bumpy” Signal

log(s)

SombreroWavelet

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Wavelets Transform

- Can be written as linear filtering

\[
W_f(\mu, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t) \overline{\Psi}(\frac{t - \mu}{s}) dt
\]

\[
= \{ f(t) \ast \overline{\Psi}_s(t) \} (\mu)
\]

\[
\overline{\Psi}_s = \frac{1}{\sqrt{s}} \Psi(\frac{t}{s})
\]

- Wavelet coefficients are a result of bandpass filtering
Wavelet Transform

• Many different constructions for different signals
  – Haar good for piece-wise constant signals
  – Battle-Lemarie’ : Spline polynomials

• Can construct Orthogonal wavelets
  – For example: dyadic Haar is orthonormal

\[ \overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi \left( \frac{t - 2^i n}{2^i} \right) \]
\( i = [1, 2, 3, \cdots] \)
Orthonormal Haar

Same scale non-overlapping

Orthogonal between scales
Scaling function

\[ \overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right) \]

• Problem:
  – Every stretch only covers half remaining bandwidth
  – Need Infinite functions

recall, for chirp:
Scaling function

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right)$$

- **Problem:**
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions
- **Solution:**
  - Plug low-pass spectrum with a scaling function $\overline{\Phi}$
Haar Scaling function

\[ \Psi(t) = \begin{cases} 
-1 & 0 \leq t < \frac{1}{2} \\
1 & \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ \Phi(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]
Back to Discrete

- Early 80’s, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80’s link to DSP by Daubechies and Mallat.

- From CWT to DWT not so trivial!
- Must take care to maintain properties
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Example: Discrete Haar Wavelet
Haar for $n=2$

Equivalent to DFT$_2$!
Discrete Orthogonal Haar Wavelet

Haar for $n=8$

scaling $\Phi_{20}$

$\Psi_{20}$

$\Psi_{10}$

$\Psi_{11}$

$\Psi_{00}$

$\Psi_{01}$

$\Psi_{02}$

$\Psi_{03}$

$\omega$

$t$
Fast DWT with Filter Banks (more Later!)

\[ h_0[n] \rightarrow h_1[n] \]

\[ x[n] \rightarrow h_0[n] \rightarrow a_{0n}? \]
\[ \quad \rightarrow h_1[n] \rightarrow d_{0n}? \]

not quite... too many coefficients
Fast DWT with Filter Banks

\[ x[n] \rightarrow h_0[n] \rightarrow h_1[n] \rightarrow d_0[n] \rightarrow \downarrow 2 \rightarrow a_0[n] \downarrow 2 \]
Fast DWT with Filter Banks

\[ x[n] \rightarrow h_0[n] \rightarrow h_1[n] \rightarrow \downarrow 2 \rightarrow a_{0n} \] \[ h_0[n] \rightarrow h_1[n] \rightarrow \downarrow 2 \rightarrow d_{0n} \]

\[ h_0[n] \rightarrow h_0[n] \rightarrow a_{1n} \]

\[ h_1[n] \rightarrow h_1[n] \rightarrow \downarrow 2 \rightarrow d_{1n} \]

Complexity:

\[ N + N/2 + N/4 + N/8 + \ldots + = 2N = O(N) \]
Decomposition

\[ x[n] \rightarrow h_0[n] \rightarrow h_1[n] \rightarrow d_0[n] \]

\[ h_0[n] \rightarrow \downarrow 2 \rightarrow a_0[n] \]

\[ h_1[n] \rightarrow \downarrow 2 \rightarrow d_0[n] \]

\[ h_0[n] \rightarrow \downarrow 2 \rightarrow a_1[n] \]

\[ h_1[n] \rightarrow \downarrow 2 \rightarrow d_1[n] \]
Reconstruction

Just flip arrows, replace $h$ with $g$
Haar DWT Example

x[n]

Haar

\[ a_{2n} \quad d_{2n} \quad d_{1n} \quad d_{0n} \]
Approximation from 25/256 coefficients

Haar

DFT
Example: Denoising Noisy Signals

Haar
Example: Denoising by Thresholding

noisy

denoised
largest 25 coefficients
Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet

Jpeg - DCT

@ 66 fold compression ratio
Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet

Jpeg - DCT

@ 66 fold compression ratio
Noisy Wavelet Denoised
Approximation/Compression

0.000% coefficients
Example in Research

Robust 4D Flow Denoising using Divergence-free Wavelet Transform

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Running head: 4D Flow Denoising with Divergence-free Wavelet Transform

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courtesy, Frank Ong and Marcus Alley
Noisy Flow Data
Divergence Free Wavelets

(a) Linear spline $\Phi_0$
  Quadratic spline $\Phi_1$

(b) Linear spline $\psi_0$
  Quadratic spline $\psi_1$

(c) Linear
  Softthresh ($\lambda n$)
  Linear
  Softthresh ($\lambda df_1$)
  Linear
  Softthresh ($\lambda df_2$)

IWT (vx)
IWT (vy)
IWT (vz)
Divergence-Free Wavelet Denoising