EE123
Digital Signal Processing

Lecture 13
DWT
• Early 80’s, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
• Late 80’s link to DSP by Daubechies and Mallat.

• From CWT to DWT not so trivial!
• Must take care to maintain properties
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Example: Discrete Haar Wavelet

Haar for $n=2$

$\Phi_{0,0}$

$\Psi_{0,0}$

scaling function

approximation

Equivalent to DFT$_2$!
Discrete Orthogonal Haar Wavelet

Haar for n=8

scaling $\Phi_{20}$

$\Psi_{20}$

$\Psi_{10}$

$\Psi_{11}$

$\Psi_{00}$

$\Psi_{01}$

$\Psi_{02}$

$\Psi_{03}$

M. Lustig, EECS UC Berkeley
Fast DWT with Filter Banks (more Later!)

\[
\begin{align*}
 h_0[n] & \rightarrow h_1[n] \\
 x[n] & \rightarrow h_0[n] \rightarrow a_{0n}? \quad \text{not quite...} \\
 & \quad \quad \text{too many coefficients} \\
 & \rightarrow h_1[n] \rightarrow d_{0n}?
\end{align*}
\]
Fast DWT with Filter Banks

\[ h_0[n] \quad h_1[n] \]

\[ x[n] \xrightarrow{h_0[n]} \xrightarrow{h_1[n]} \xrightarrow{\downarrow 2} \]

\[ \xrightarrow{\downarrow 2} a_{0n} \quad \xrightarrow{\downarrow 2} d_{0n} \]
Fast DWT with Filter Banks

\[ h_0[n] \rightarrow h_1[n] \]

complexity:
\[ N + N/2 + N/4 + N/8 + \ldots + = 2N \]
\[ = O(N) \]

\( x[n] \)
\( h_0[n] \rightarrow \downarrow 2 \rightarrow a_{0n} \)
\( h_1[n] \rightarrow \downarrow 2 \rightarrow d_{0n} \)

\( h_0[n] \rightarrow \downarrow 2 \rightarrow a_{1n} \)
\( h_1[n] \rightarrow \downarrow 2 \rightarrow d_{1n} \)
Decomposition

\[ x[n] \rightarrow h_0[n] \rightarrow \downarrow 2 \rightarrow a_{0n} \]

\[ x[n] \rightarrow h_1[n] \rightarrow \downarrow 2 \rightarrow d_{0n} \]
Reconstruction

\[ x[n] \leftarrow g_0[n] \leftarrow \uparrow 2 \text{ a}_{0n} \]

\[ g_1[n] \leftarrow \uparrow 2 \text{ d}_{0n} \]

Just flip arrows, replace h with g
Example, Haar DWT - Level 0
Example, Haar DWT - Level 1
Example, Haar DWT - Level 2
Example, Haar DWT - Level 3
Example, Haar DWT - Level 4
Example, Haar DWT - Level 5
DWT Another view
Haar DWT Example

The diagram illustrates the Haar Discrete Wavelet Transform (DWT) for a given signal $x[n]$. The graph shows the approximation coefficients $a_{2n}$ and the detail coefficients $d_{2n}$, $d_{1n}$, and $d_{0n}$.
Approximation from 25/256 coefficients

**Haar**

![Haar Approximation](image)

**DFT**

![DFT Approximation](image)
Example: Denoising Noisy Signals

Haar
Example: Denoising by Thresholding

noisy

denoised

largest 25 coefficients
Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet

Jpeg - DCT

@ 66 fold compression ratio
Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet
Jpeg - DCT

@ 66 fold compression ratio
Approximation/Compression
Example in Research

Robust 4D Flow Denoising using Divergence-free Wavelet Transform

Frank Ong\textsuperscript{1}, Martin Uecker\textsuperscript{1}, Umar Tariq\textsuperscript{2}, Albert Hsiao\textsuperscript{2}, Marcus T Alley\textsuperscript{2}, Shreyas S Vasanawala\textsuperscript{2}, Michael Lustig\textsuperscript{1}

courtesy, Frank Ong and Marcus Alley
Noisy Flow Data

- DivFree & Non-DivFree Thresh Non-DivFree Thresh Only
- Original DivFree Wavelet Manual Threshold

Emitter plane from descending aorta
Emitter plane from ascending aorta
Emitter plane from ascending aorta
Original DFW Manual Threshold

Vector visualization Streamline visualization

0 cm/s 160 cm/s
0 cm/s 100 cm/s
Divergence Free Wavelets

(a) Linear spline $\Phi_0$  
Quadratic spline $\Phi_1$  
Linear spline $\psi_0$  
Quadratic spline $\psi_1$

(b) 

(c)
Divergence-Free Wavelet Denoising

- DivFree & Non-DivFree Thresh
- Non-DivFree Thresh Only

Original

DFW Manual Threshold

Vector visualization Streamline visualization

Emitter plane from descending aorta

Analysis plane from ascending aorta