Lecture 14
Sampling
Announcements

• Ham exam Th 3/12 7-10+, The Woz Soda hall
• Lab:
  – Who is having trouble?
What is this Phenomena?
Sampling of Continuous Time Signals (Ch. 4)

• **Sampling:**
  – Conversion from C.T (not quantized) into D.T (usually quantized)

• **Reconstruction**
  – D.T (quantized) to C.T

• **Why?**
  – Digital storage (audio, images, videos)
  – Digital communications (fiber optics, cellular...)
  – DSP (compression, correction, restoration)
  – Digital synthesis (speech, graphics)
  – Learning
Sampling of C.T. Signals

• Typical System:

\[ x_c(t) \quad \text{Analog Anti-Aliasing Filter} \quad x[n] = x_c(nT) \quad \text{Quantizer} \]

\[ t = nT \]

\[ y[n] \quad \text{Discrete stuff (DSP, storage....)} \quad y_c(t) \quad \text{Reconstruction} \]

DAC D/A

ADC A/D

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Ideal Sampling Model

\[ x_c(t) \xrightarrow{C/D} x[n] = x_c(nT) \]

Discrete and Continuous

define impulsive sampling:

\[ x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots \]
\[ x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT) \]
Ideal Sampling Model

\[ x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

- Not physical: used for modeling & derivations

\[ x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t) \]

- How is \( x[n] \) related to \( x_s(t) \) in freq. domain?
Frequency Domain Analysis

• How is $x[n]$ related to $x_s(t)$ in the Freq. Domain?

$$x_s(t) : \text{C.T}$$

$$X_s(j\Omega) = \sum_n x_c(nT)e^{-j\Omega nT}$$

$$x[n] : \text{D.T}$$

$$X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}$$

$$\omega = \Omega T$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega = \omega/T}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega = \Omega T}$$
Frequency Domain Analysis

• How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \sum_n \delta(t - nT)$$

$\triangleq s(t)$
Frequency Domain Analysis

• How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \sum_{n} \delta(t - nT)$$

$\triangleq s(t)$

recall $\mathcal{W}(t) = \sum_n \delta(t-n)$

$$s(t) = \sum_n \delta(t-nT) = \sum_n \delta(T(\frac{t}{T}-n)) = \frac{1}{T} \sum_n \delta(\frac{t}{T}-n) = \frac{1}{T} \mathcal{W}(\frac{t}{T})$$

notation Break

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Frequency Domain Analysis

• How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \sum_{n} \delta(t - nT) \triangleq s(t)$$

$s(t) \leftrightarrow S(j\Omega)$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

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Frequency Domain Analysis

\[ X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) \ast S(j\Omega) \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - \Omega_s)) \quad \mid \quad \Omega_s = \frac{2\pi}{T} \]

• \( X_s \) is replication of \( X_c \)!
Frequency Domain Analysis

So, if:

\[ X_c(j\Omega) \quad \text{and} \quad \Omega_s > 2\Omega_N \]

\[ \omega = \Omega T \]

\[ \frac{\Omega_s}{2} \cdot T = \pi \]
Aliasing

So, if:

\[ X_c(j\Omega) \quad \text{and} \quad \Omega_s < 2\Omega_N \]

\[ \omega = \Omega T \]

\[ X_s(j\Omega) \]

\[ \frac{\Omega_s}{2} \cdot T = \pi \]
Aliasing

Q: What is the difference in acquisition between the two images?
\[ \Delta k = \frac{1}{\text{FOV}} \]
Design:
abdominal MRI will be recruited.

Subjects:
in the localizer and branch hepatic artery. In the images show aliased peripheral IV tubing with true position shown.
mesenteric veins and liver capsule. Zoomed insets in images show left gastric artery.

SPIRiT compressed resolution images are too noisy. Images show decreased noise and improved structure delineation. Note improved delineation of specific anatomic structures over standard methods.

Figure 36:

Figure 37:

Each patient will undergo an MRI protocol as shown in.

Table 2:

MRI protocol to validate new developments in DnD Tl sequences with.

Table:

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Reconstruction of Bandlimited Signals

• Nyquist Sampling Thm: suppose $x_c(t)$ is bandlimited

$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

if $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT')$

• Bandlimitedness is the key to uniqueness

multiple signals go through the samples, but only one is bandlimited!
Reconstruction in Frequency Domain

\[ x[n] \xrightarrow{\text{Convert to impulse train}} x_s(t) \xrightarrow{H_r(j\Omega)} x_r(t) \]

\[ X_s(j\Omega) \]

\[ H_r(j\Omega) \]

\[ X_r(j\Omega) \]
Reconstruction in Time Domain

\[ h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \]

\[ = \frac{T}{2\pi} \left. \frac{1}{j} \frac{e^{j\Omega t}}{\Omega} \right|_{-\Omega_s/2}^{\Omega_s/2} \]

\[ = \frac{T}{\pi t} \left( e^{j\frac{\Omega_s}{2} t} - e^{-j\frac{\Omega_s}{2} t} \right) \]

\[ = \frac{T}{\pi t} \sin(\frac{\Omega_s t}{2}) = \frac{T}{\pi t} \sin(\frac{\pi}{T} t) \]

\[ = \text{sinc}(\frac{t}{T}) \]
Reconstruction in Time Domain

\[ x_r(t) = x_s(t) \star h_r(t) = \left( \sum_n x[n] \delta(t - nT) \right) \star h_r(t) \]

\[ = \sum_n x[n] h_r(t - nT) \]

The sum of “sincs gives \( x_r(t) \Rightarrow \text{Unique signal} \)

bandlimited by \( \Omega_s \)
• If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

\[
X_r(j\Omega) = \begin{cases} 
TX_s(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\
0 & \text{otherwise}
\end{cases}
\]
Anti-Aliasing

$x_c(t)$

Analog Anti-Aliasing Filter $H_{LP}(j\Omega)$

$sampler$

$X_c(j\Omega)$ and $\Omega_s < 2\Omega_N$

$X_s(j\Omega)$ and $\Omega_s < 2\Omega_N$

$X_c(j\Omega)H_{LP}(j\Omega)$ and $\Omega_s < 2\Omega_N$

$X_s(j\Omega)$ and $\Omega_s < 2\Omega_N$
Non Ideal Anti-Aliasing

\[ X_c(j\Omega)H_{LP}(j\Omega) \]

interference

\[-\Omega_N \quad \Omega_N \]

\[ \Omega_s/2 \]

\[ X(e^{j\Omega}) \]

\[-\pi \quad \pi \]
SDR non-perfect anti-Aliasing Demo