Topics

• Did you sign up for the ham exam?
• Last time
  – D.T processing of C.T signals
  – C.T processing of D.T signals (ha??????)
  – Downsampling
• Today
  – Changing Sampling Rate via DSP
  – Upsampling
  – Rational resampling
  – Interchanging operations
Review DownSampling

• Much like C/D conversion
• Expect similar effects:
  – Aliasing
  – mitigate by antialiasing filter

• Finely sampled signal \(\Rightarrow\) almost continuous
  – Downsample in that case is like sampling!
Changing Sampling-rate via D.T Processing

\[ X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{\infty} X \left( e^{j(\omega/M - 2\pi i / M)} \right) \]
Changing Sampling-rate via D.T Processing

\[ X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{\infty} X \left( e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)} \right) \]
Anti-Aliasing

\[ x[n] \xrightarrow{\text{LPF}} \tilde{x}[n] \xrightarrow{\downarrow M} \tilde{x}_d[n] = \tilde{x}[nM] \]
UpSampling

- Much like D/C converter
- Upsample by A LOT ⇒ almost continuous

- Intuition:
  - Recall our D/C model: \( x[n] \mapsto x_s(t) \mapsto x_c(t) \)
  - Approximate “\( x_s(t) \)” by placing zeros between samples
  - Convolve with a sinc to obtain “\( x_c(t) \)”
Up-sampling

\[ x[n] = X_c(nT) \]

\[ x_i[n] = X_c(nT') \quad \text{where} \quad T' = \frac{T}{L} \quad L \text{ integer} \]

Obtain \( x_i[n] \) from \( x[n] \) in two steps:

(1) Generate: \( x_e = \begin{cases} 
  x[n/L] & n = 0, \pm L, \pm 2L, \cdots \\
  0 & \text{otherwise}
\end{cases} \)
(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

$$x_i = x_e[n] * \text{sinc}\left(\frac{n}{L}\right)$$
Up-Sampling

\[ x_i[n] = x_e[n] * \text{sinc}(n/L) \]

\[ x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \]

\[ x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \text{sinc}(\frac{n - kL}{L}) \]
Frequency Domain Interpretation

\[ x[n] \rightarrow \uparrow L \rightarrow x_e[n] \rightarrow \text{LPF} \begin{array}{c} \text{gain}=L \\ \pi/L \end{array} \rightarrow x_i[n] \]

\[ \text{sinc}(n/L) \quad \text{DTFT} \Rightarrow \]

\[ -\frac{\pi}{L} \quad \frac{\pi}{L} \]
Frequency Domain Interpretation

\[ x[n] \xrightarrow{\uparrow L} x_e[n] \xrightarrow{\text{LPF}} x_i[n] \]

\[ X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_e[n] e^{-j\omega n} \]

\[ = \sum_{m=-\infty}^{\infty} x_e[mL] e^{-j\omega mL} \]

\[ = x[m] \]

\[ = X(e^{j\omega L}) \]

Compress DTFT by a factor of L!
Example:

\[ X_c(j\Omega) \]

- **Sampling** \( T \)
- **Expanding** \( L \)

\[ X(e^{j\omega}) \]

\[ X_i(e^{j\omega}) \]

\[ X_e(e^{j\omega}) \]

\[ \Omega_N \]

\[ \Omega \]
Example:

\[ X(c(j\Omega)) \]

Sampling \( T \)

\[ X(e^{j\omega}) \]

Expanding \( L \)

\[ X_e(e^{j\omega}) \]

Sampling \( T'=T/L \)
Example:

$X_c(j\Omega)$

$X(e^{j\omega})$

$X_i(e^{j\omega})$

$X_e(e^{j\omega})$

sampling $T$

expanding $L$

sampling $T' = T/L$
Example:

- $X_c(j\Omega)$
- $X(e^{j\omega})$
- $X_i(e^{j\omega})$
- $X_e(e^{j\omega})$

**Sampling T:**

- $X(e^{j\omega})$ in the frequency domain.

**Sampling T' = T/L:**

- $X_i(e^{j\omega})$ in the frequency domain.

**Expanding L:**

- $X_e(e^{j\omega})$ in the frequency domain.
Example:

$$X_e(e^{j\omega})$$

$$X_i(e^{j\omega})$$

$$X_c(j\Omega)$$

sampling $T$

sampling $T'=T/L$

expanding $L$
Practical Upsampling

- Can interpolate with simple, practical filters. What’s happening?
- Example: $L=3$, linear interpolation - convolve with triangle

\[ x[n] \]
Resampling by non-integer

- $T' = TM/L$ (upsample $L$, downsample $M$)

Or,

- What would happen if change order?

$\min\{\pi/L, \pi/M\}$
Example:

- $L = 2$, $M = 3$, $T' = \frac{3}{2}T$ (fig 4.30)

$X_c(j\Omega)$

Sampling $T$

$X(e^{j\omega})$

Expanding $L = 2$

Subsampling $M = 3$

$X_e(e^{j\omega})$

LP filtering

$\tilde{X}_i = H_d X_e$
Example:

• $L = 2, M = 3, \quad T' = \frac{3}{2}T$ (fig 4.30)

\[ X_c(j\Omega) \]

\[ X(e^{j\omega}) \]

\[ X_e(e^{j\omega}) \]

\[ \tilde{X}_i = H_d X_e \]
Multi-Rate Signal Processing

• What if we want to resample by $1.01T$?
  – Expand by $L=100$
  – Filter $\pi/101$ ($\cdots$)
  – Downsample by $M=101$

• Fortunately there are ways around it!
  – Called multi-rate
  – Uses compressors, expanders and filtering
Interchanging Operations

Note:

\[ H(e^{j\omega})X(e^{j\omega}) \]

\[ H(e^{j\omega L})X(e^{j\omega L}) \]

\[ X(e^{j\omega L})H(e^{j\omega})X(e^{j\omega L}) \]
Interchanging Operations

\[ H(z) \uparrow L \quad \text{“expander”} \quad \rightarrow \quad \downarrow M \quad \text{“compressor”} \]

Note:

\[ x[n] \rightarrow H(z) \uparrow L \rightarrow y[n] \neq x[n] \uparrow L \rightarrow H(z) \rightarrow y[n] \]

\[ H(e^{j\omega})X(e^{j\omega}) \]

\[ H(e^{j\omega L})X(e^{j\omega L}) \]

\[ X(e^{j\omega L})H(e^{j\omega})X(e^{j\omega L}) \]

\[ \equiv x[n] \uparrow L \rightarrow H(z^L) \rightarrow y[n] \]

\[ X(e^{j\omega L})H(e^{j\omega L})X(e^{j\omega L}) \]

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Q: Can we move expander from Left to Right (with xform)?

A: Yes, if $H(z)$ is rational
No, otherwise
Example:

\[ (2^2) = 4 - 2^2 \text{ not rational} \]

This can't be written as output of expander
(every other value is not zero)