Topics

• Did you sign up for the ham exam?
• Last time
  – D.T processing of C.T signals
  – C.T processing of D.T signals (ha??????)
  – Downsampling
• Today
  – Changing Sampling Rate via DSP
  – Upsampling
  – Rational resampling
  – Interchanging operations
Review DownSampling

• Much like C/D conversion
• Expect similar effects:
  – Aliasing
  – mitigate by antialiasing filter

• Finely sampled signal $\Rightarrow$ almost continuous
  – Downsample in that case is like sampling!
Changing Sampling-rate via D.T Processing

\[ X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left( e^{j(\omega/M - 2\pi i/M)} \right) \]
Changing Sampling-rate via D.T Processing

\[ X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0} X \left( e^{j(w/M - 2\pi i / M)} \right) \]

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Anti-Aliasing

\[ x[n] \xrightarrow{\text{LPF } \pi/M} \tilde{x}[n] \xrightarrow{\downarrow M} \tilde{x}_d[n] = \tilde{x}[nM] \]

\[ X \]

\[ X_d \]

\[ M=3 \]
UpSampling

• Much like D/C converter
• Upsample by A LOT ⇒ almost continuous

• Intuition:
  – Recall our D/C model: $x[n] \Rightarrow x_s(t) \Rightarrow x_c(t)$
  – Approximate “$x_s(t)$” by placing zeros between samples
  – Convolve with a sinc to obtain “$x_c(t)$”
Up-sampling

\[ x[n] = X_c(nT) \]

\[ x_i[n] = X_c(nT') \quad \text{where} \quad T' = \frac{T}{L} \quad L \text{ integer} \]

Obtain \( x_i[n] \) from \( x[n] \) in two steps:

(1) Generate:

\[ x_e = \begin{cases} 
  x[n/L] & n = 0, \pm L, \pm 2L, \cdots \\
  0 & \text{otherwise}
\end{cases} \]
Up-Sampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

$$x_i = x_e[n] \ast \text{sinc} \left( \frac{n}{L} \right)$$
Up-Sampling

\[ x_i[n] = x_e[n] * \text{sinc}(n/L) \]

\[ x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \]

\[ x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \text{sinc}\left(\frac{n - kL}{L}\right) \]
Frequency Domain Interpretation

$$x[n] \xrightarrow{\uparrow L} x_e[n] \xrightarrow{\text{LPF gain}=L \frac{\pi}{L}} x_i[n]$$

$$\text{sinc}(n/L) \quad \text{DTFT} \Rightarrow$$

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Frequency Domain Interpretation

\[ X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_e[n] e^{-j\omega n} \]

\( \neq 0 \) only for \( n=mL \) (integer \( m \))

\[ = \sum_{m=-\infty}^{\infty} x_e[mL] e^{-j\omega mL} = X(e^{j\omega L}) \]

Compress DTFT by a factor of \( L \)!
Example:

\[ X_c(j\Omega) \]

\[ X(e^{j\omega}) \]

\[ X_i(e^{j\omega}) \]

\[ X_e(e^{j\omega}) \]

sampling $T$

sampling $T' = T/L$

expanding $L$

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Example:

\[ X_c(j\Omega) \]

sampling T

\[ X(e^{j\omega}) \]

expanding L

\[ X_e(e^{j\omega}) \]

sampling T’=T/L

\[ X_i(e^{j\omega}) \]
Example:

\[ X_c(j\Omega) \]

\[ X(e^{j\omega}) \]

\[ X_i(e^{j\omega}) \]

\[ X_e(e^{j\omega}) \]

sampling \( T \)

sampling \( T' = T/L \)

expanding \( L \)
Example:

\[ X_c(j\Omega) \]

**sampling T**

\[ X(e^{j\omega}) \]

**expanding L**

\[ X_e(e^{j\omega}) \]

**sampling T’=T/L**

\[ X_i(e^{j\omega}) \]

\[ \cdots \]

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Example:

\[ X_c(j\Omega) \]

\[ X_e(e^{j\omega}) \]

\[ X_i(e^{j\omega}) \]

\[ X(e^{j\omega}) \]

- Sampling \( T \)
- Sampling \( T' = T/L \)
- Expanding \( L \)

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Practical Upsampling

- Can interpolate with simple, practical filters. What’s happening?
- Example: L=3, linear interpolation - convolve with triangle

\[
x[n]
\]

\[
\frac{1}{T}
\]

\[
\frac{3}{\pi}
\]

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Resampling by non-integer

• $T' = TM/L$ (upsample $L$, downsample $M$)

Or,

• What would happen if change order?
Example:

- $L = 2, M = 3, \quad T' = 3/2T$ (fig 4.30)

$$X_c(j\Omega)$$

1

$\Omega_N$  $\Omega$

sampling $T$

$X(e^{j\omega})$

$X_e(e^{j\omega})$

expanding $L = 2$

Subsampling $M = 3$

LP filtering

$$\tilde{X}_i = H_d X_e$$

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Example:

- $L = 2$, $M = 3$, $T' = 3/2T$ (fig 4.30)

$$X_c(j\Omega)$$

Sampling $T$

$$X(e^{j\omega})$$

Subsampling $M = 3$

Expanding $L = 2$

LP filtering

$$\tilde{X}_i = H_dX_e$$
Multi-Rate Signal Processing

• What if we want to resample by $1.01T$?
  – Expand by $L=100$
  – Filter $\pi/101$ ($\$$\$$\$$\$$)$
  – Downsample by $M=101$

• Fortunately there are ways around it!
  – Called multi-rate
  – Uses compressors, expanders and filtering
Interchanging Operations

 Volunteers 

 "expander" 

 "compressor" 

 not LTI!

 Note:

 $x[n] \rightarrow H(z) \rightarrow \uparrow L \rightarrow y[n]$ 

 $H(e^{j\omega})X(e^{j\omega})$

 $H(e^{j\omega L})X(e^{j\omega L})$

 $x[n] \rightarrow H(z) \rightarrow \uparrow L \rightarrow y[n]$ 

 $X(e^{j\omega L}) \rightarrow H(e^{j\omega}) \rightarrow X(e^{j\omega L})$
Interchanging Operations

“expander”

“compressor”

Note:

\[
x[n] \rightarrow H(z) \uparrow L \rightarrow y[n] \neq x[n] \rightarrow \uparrow L \rightarrow H(z) \rightarrow y[n]
\]

\[
H(e^{j\omega})X(e^{j\omega}) = H(e^{j\omega L})X(e^{j\omega L})
\]

\[
X(e^{j\omega L}) = H(e^{j\omega})X(e^{j\omega L})
\]

\[
\equiv x[n] \rightarrow \uparrow L \rightarrow H(z^L) \rightarrow y[n]
\]

\[
X(e^{j\omega L}) = H(e^{j\omega L})X(e^{j\omega L})
\]
Interchanging Filter Expander

• Q: Can we move expander from Left to Right (with xform)?

$$H(z) \uparrow L \equiv? \uparrow L$$

• A: Yes, if $H(z)$ is rational
  No, otherwise
Example:

\[ H(z^2) = H(z)z^{-2} \text{ not rational} \]

this can't be written as output of an expander
(every other value is not zero)

\[ H(z^2) = H(z)z^{-2} \]

every odd value is zero