Lecture 17
Lab III
Polyphase Filters
Topics

• Last time
  – Changing Sampling Rate via DSP
  – Upsampling
  – Rational resampling

• Today
  – Lab III
  – Interchanging Compressors/Expanders and filtering
  – Polyphase decomposition
  – Multi-rate processing
Lab III - Time-Frequency

- compute spectrograms with w/o windowing
Lab III - Time-Frequency

- Compute with overlapping window
Lab III - Time-Frequency

• Look at temporal/frequency resolution tradeoffs:
FM Broadcast Radio - KPFA 94.1MHz
Spectrogram of Broadcast FM

Non-demodulated

FM demodulated
Spectrogram of Broadcast FM

FM demodulated

Broadcast FM baseband signal

L+R (mono)

pilot signal

L-R (stereo)

RBDS

Subcarriers

19KHz  38KHz  57KHz  67.65KHz  92KHz
Filter Mono and down

- To play we need to filter the right signal
- Downsampling to 48KHz so we can play on the computer
Demodulate subcarriers: Example 92KHz

demodulate by 92KHz
Demodulate subcarriers: Example 92KHz

- Filter and decimate

- FM demodulate and filter
Multi-Rate Signal Processing

• What if we want to resample by $1.01T$?
  – Expand by $L=100$
  – Filter $\pi/101$ ($\$$$$$
  – Downsample by $M=101$

• Fortunately there are ways around it!
  – Called multi-rate
  – Uses compressors, expanders and filtering
Interchanging Operations

Not LTI!

Note:

$$x[n] \rightarrow H(z) \rightarrow \uparrow L \rightarrow y[n]$$

$$x[n] \rightarrow \uparrow L \rightarrow H(z) \rightarrow y[n]$$

$$H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega L})X(e^{j\omega L})$$

$$X(e^{j\omega L})H(e^{j\omega})X(e^{j\omega L})$$
Interchanging Operations

Note:

\[
x[n] \rightarrow H(z) \rightarrow \uparrow L \rightarrow y[n] \neq x[n] \rightarrow \uparrow L \rightarrow H(z) \rightarrow y[n]
\]

\[
H(e^{j\omega})X(e^{j\omega}) \quad \quad X(e^{j\omega L})H(e^{j\omega})X(e^{j\omega L})
\]

\[
\equiv \quad x[n] \rightarrow \uparrow L \rightarrow H(z^L) \rightarrow y[n]
\]

\[
X(e^{j\omega L}) \quad \quad H(e^{j\omega L})X(e^{j\omega L})
\]

not LTI!
Interchanging Filter Expander

• Q: Can we move expander from Left to Right (with xform)?

- A: Yes, if $H(z)$ is rational
  No, otherwise
Compressor

Claim:

\[ x[n] \downarrow M \rightarrow H(z) \rightarrow y[n] \equiv x[n] \rightarrow H(z^M) \downarrow M \rightarrow \tilde{y}[n] \]

Proof:
Compressor

Proof:

\[ Y(e^{i\omega}) = H(e^{j\omega}) \left( \frac{1}{M} \sum_{i=0}^{M-1} X \left( e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)} \right) \right) \]

\[ = \frac{1}{M} \sum_{i=0}^{M-1} H \left( e^{j(\omega - 2\pi i)} \right) X \left( e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)} \right) \]

\[ = \frac{1}{M} \sum_{i=0}^{M-1} H \left( e^{jM\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)} \right) X \left( e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)} \right) \]

\[ V(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega}) \]

after compressor
Compressor

Claim:

\[ x[n] \rightarrow \downarrow M \rightarrow H(z) \rightarrow y[n] \equiv x[n] \rightarrow H(z^M) \rightarrow \downarrow M \rightarrow \tilde{y}[n] \]

Proof:

\[
Y(e^{j\omega}) = H(e^{j\omega}) \left( \frac{1}{M} \sum_{i=0}^{M-1} X \left( e^{j \left( \frac{\omega}{M} - \frac{2\pi i}{M} \right)} \right) \right)
\]

\[
= \frac{1}{M} \sum_{i=0}^{M-1} H \left( e^{j(\omega-2\pi i)} \right) X \left( e^{j \left( \frac{\omega}{M} - \frac{2\pi i}{M} \right)} \right) \text{ after compressor}
\]

Q: Move Compressor from right to left?
A: Only if \( H(z^{1/M}) \) is rational!

\[
V(e^{j\omega}) = H(e^{j\omega M}) X(e^{j\omega})
\]
Interchanging Operations

\[
\begin{align*}
x[n] & \rightarrow H(z) \rightarrow \uparrow L \rightarrow y[n] \\
\Downarrow M \rightarrow H(z) & \rightarrow y[n] \\
\end{align*}
\]

\[
\begin{align*}
x[n] & \rightarrow \uparrow L \rightarrow H(z^L) \rightarrow y[n] \\
\Downarrow M \rightarrow H(z^M) & \rightarrow \downarrow M \rightarrow y[n] \\
\end{align*}
\]
Polyphase Decomposition

• We can decomposed an impulse response to:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]
Polyphase Decomposition

• Define:

\[ h_k[n] \rightarrow \downarrow M \rightarrow e_k[n] \]

\[ e_k[n] = h_k[nM] \]
Polyphase Decomposition

\[ e_k[n] \xrightarrow{\uparrow M} h_k[n] \]

recall upsampling \(\Rightarrow\) scaling

\[ H_k(z) = E_k(z^M) \]

Also, recall:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]

So,

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \]
Polyphase Decomposition

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k} \]

Why should you care?
Polyphase Implementation of Decimation

\[ x[n] \xrightarrow{H(z)} y[n] \xrightarrow{\downarrow M} w[n] = y[nM] \]

• Problem:
  – Compute all \( y[n] \) and then throw away -- wasted computation!
    • For FIR length \( N \Rightarrow N \) mults/unit time
  – Can interchange Filter with compressor?
    • Not in general!
Polyphase Implementation of Decimation

\[
x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM]
\]
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]

Interchange sum with decimation

\[ x[n] \rightarrow E_0(z^M) \rightarrow \downarrow M \rightarrow E_1(z^M) \rightarrow \downarrow M \rightarrow \ldots \rightarrow E_{M-1}(z^M) \rightarrow \downarrow M \rightarrow + \rightarrow w[n] \]

now, what can we do?
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]

Interchange filter with decimation

Computation:
Each Filter: \( N/M \times (1/M) \) mult/unit time
Total: \( N/M \) mult/unit time

what about interpolation?