Lecture 18
Filter Banks
Announcements

- Lab III due Sunday 11:55pm
- HW6 due Monday 11:55pm
- Midterm II Rescheduled Options 3/31 6:30-8:30 or 4/3 2-4 or 3-5
- Ham radio exam this Thursday (!!!)
  - Get your licenses next week
  - Get radios when you get a callsign
Last Time

• Polyphase decomposition

Today:

– Multi-rate Filter Banks
– Subtleties in Time-Frequency tiling
– Perfect reconstruction with non-ideal filters
– Polyphase filter banks
Polyphase Decomposition

- We can decomposed an impulse response to:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]
Polyphase Decomposition

• Define:

\[ h_k[n] \rightarrow \downarrow M \rightarrow e_k[n] \]

\[ e_k[n] = h_k[nM] \]

\[ h_0[n] \]

\[ h_1[n] \]

\[ e_0[n] \]

\[ e_1[n] \]
Polyphase Decomposition

\[ e_k[n] \rightarrow \uparrow M \rightarrow h_k[n] \]

recall upsampling ⇒ scaling

\[ H_k(z) = E_k(z^M) \]

Also, recall:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]

So,

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \]
Polyphase Decomposition

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \]

Why should you care?
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \downarrow M \rightarrow w[n] = y[nM] \]

• Problem:
  – Compute all \( y[n] \) and then throw away -- wasted computation!
    • For FIR length \( N \Rightarrow N \) mults/unit time
  – Can interchange Filter with compressor?
    • Not in general!
Polyphase Implementation of Decimation

\[ x[n] \xrightarrow{H(z)} y[n] \xrightarrow{\downarrow M} w[n] = y[nM] \]

\[ z^{-1} \]

\[ H(z) \]

\[ x[n] \]

\[ E_0(z^M) \]

\[ E_1(z^M) \]

\[ \cdots \]

\[ E_{M-1}(z^M) \]

\[ + \]

\[ y[n] \xrightarrow{\downarrow M} w[n] \]
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]

Interchange filter with decimation

\[ x[n] \rightarrow E_0(z^M) \rightarrow \downarrow M \]
\[ \rightarrow E_1(z^M) \rightarrow \downarrow M \]
\[ \rightarrow \ldots \]
\[ \rightarrow E_{M-1}(z^M) \rightarrow \downarrow M \]
\[ \rightarrow + \rightarrow w[n] \]

now, what can we do?
Polyphase Implementation of Decimation

\[ x[n] \xrightarrow{H(z)} y[n] \xrightarrow{\downarrow M} w[n] = y[nM] \]

Interchange filter with decimation

\[ x[n] \xrightarrow{\downarrow M} \xrightarrow{z^{-1}} \xrightarrow{\downarrow M} \xrightarrow{z^{-1}} \xrightarrow{\downarrow M} \xrightarrow{z^{-1}} \]

Computation:
Each Filter: \( \frac{N}{M} \times \frac{1}{M} \) mult/unit time
Total: \( \frac{N}{M} \) mult/unit time

what about interpolation?
Multirate FilterBank

- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
- Often $h_1[n] = e^{j\pi n} h_0[n]$ or $H_1(e^{j\omega}) = H_0(e^{j(\omega - \pi)})$
Subtleties in Time-Freq Tiling

• Assume $h_0, h_1$ are ideal low, high pass filters
Subtleties in Time-Freq Tiling

• Assume $h_0, h_1$ are ideal low, high pass filters.

\[
\begin{align*}
  x[n] & \xrightarrow{h_0[n]} \downarrow 2 \xrightarrow{h_1[n]} \downarrow 2 \\
  X(e^{j\omega}) & \text{Frequency domain representation}
\end{align*}
\]
Subtleties in Time-Freq Tiling

- Assume $h_0, h_1$ are ideal low, high pass filters

Mathematical expressions and diagrams illustrating the process of signal processing through ideal low and high pass filters, followed by downsampling operations.
Subtleties in Time-Freq Tiling

- Assume $h_0$, $h_1$ are ideal low, high pass filters
Perfect Reconstruction Ideal Filters

\[ \mathcal{G}_0[n] \quad \mathcal{G}_1[n] \quad y[n] \]

\[ 2 \uparrow \longrightarrow \mathcal{G}_0[n] \quad \mathcal{G}_1[n] \longrightarrow 2 \uparrow \quad y[n] \]
Non ideal LP and HP Filters

\[ x[n] \rightarrow h_0[n] \rightarrow \downarrow 2 \rightarrow h_1[n] \rightarrow \downarrow 2 \rightarrow X(e^{j\omega}) \]

\[ H_0(e^{j\omega}) \quad H_1(e^{j\omega}) \]
Perfect Reconstruction non-Ideal Filters

\[ x[n] \xrightarrow{h_0[n]} \downarrow 2 \xrightarrow{h_1[n]} \downarrow 2 \xrightarrow{\text{Stuff}} 2 \uparrow \xrightarrow{2 \uparrow} g_0[n] \]

\[ X(e^{j\omega}) \xrightarrow{\text{analysis}} \cdot G_0(e^{j\omega}) \xrightarrow{\text{synthesis}} y[n] \]

\[ = X(e^{j\omega}) + \cdot G_1(e^{j\omega}) \]
Perfect Reconstruction non-Ideal Filters

\[ Y(e^{j\omega}) = \frac{1}{2} \left[ G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega}) \right] X(e^{j\omega}) \]

\[ + \frac{1}{2} \left[ G_0(e^{j\omega})H_0(e^{j(\omega - \pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega - \pi)}) \right] X(e^{j(\omega - \pi)}) \]

\text{need to cancel!}

M. Lustig, EECS UC Berkeley
Quadrature Mirror Filters - perfect recon

\[
H_1(e^{j\omega}) = H_0(e^{j(\omega - \pi)})
\]
\[
G_0(e^{j\omega}) = 2H_0(e^{j\omega})
\]
\[
G_1(e^{j\omega}) = -2H_1(e^{j\omega})
\]
Quadrature Mirror Filters - perfect recon

\[ x[n] \xrightarrow{h_0[n], h_1[n]} \xrightarrow{\downarrow 2} \text{Stuff} \xrightarrow{\uparrow 2} y[n] \]

\[ H_1(e^{j\omega}) = H_0(e^{j(\omega - \pi)}) \]
\[ G_0(e^{j\omega}) = 2H_0(e^{j\omega}) \]
\[ G_1(e^{j\omega}) = -2H_1(e^{j\omega}) \]

Example Haar:

\[ h_0[n] \xrightarrow{} g_0[n] \]
\[ h_1[n] \xrightarrow{} g_1[n] \]
\begin{align*}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n + 1] \\
e_{10} &= h_1[2n] = e^{j2\pi n} h_0[2n] = e_{00}[n] \\
e_{11} &= h_1[2n + 1] = e^{j2\pi n} e^{j\pi} h_0[2n + 1] = -e_{01}[n]
\end{align*}
Polyphase Filter-Bank

Analysis

\[ x[n] \xrightarrow{z^{-1}} \]

\[
\begin{align*}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n + 1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{align*}
\]
### Polyphase Filter-Bank

**Analysis**

- $x[n]$ is input.
- $z^{-1}$ is a delay operator.
- $e_{00}[n]$, $e_{01}[n]$, $e_{10}[n]$, $e_{11}[n]$ are output signals.

**Equations:**

- $e_{00} = h_0[2n]$
- $e_{01} = h_0[2n + 1]$
- $e_{10} = e_{00}[n]$
- $e_{11} = -e_{01}[n]$