Lecture 21
Tomography
Impulse lines and line-integrals

In 1D
\[ \int_{-\infty}^{\infty} f(x) \delta(x) \, dx = f(0) \]

A sample @x=0

In 2D
\[ \int_{-\infty}^{\infty} f(x, y) \delta(x) \, dx = f(0, y) \]

1D cross-section @x=0

Line Integral
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x) \, dy \, dx = \int_{-\infty}^{\infty} f(0, y) \, dy \]

Integral of the 1D cross-section @x=0
Impulse lines and line-integrals

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x) \, dx \, dy = f(0, y) = \nabla(y) \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - 1) \, dx \, dy = f(1, y) = \nabla(2y) \]

what about line integral with \( \delta(x-u) \)?
Line Integral and Projection

\[ p(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x-u) \, dx \, dy = \int_{-\infty}^{\infty} f(u, y) \, dy \]
General Projections

\[ p(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(\rho - x \cos \theta - y \sin \theta) \, dx \, dy \]
Many Projections - Tomography

http://www.youtube.com/watch?v=4gkIQHM19aY&feature=related
Radon Transform

\[ p(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(\rho - x \cos \theta - y \sin \theta) \, dx \, dy \]
Radon Transform: Sinogram

• Also called Sinogram
• Impulse $\Rightarrow$ Sinusoid
Computed Tomography

Sinogram

cross-section

x-ray source

FBP
Projection Slice Theorem (Bracewell)

\[ \mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta) \]
Projection Slice Theorem (Bracewell)

$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$

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Projection Slice Theorem (Bracewell)

\[ \mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta) \]
Projection Slice Theorem (Bracewell)

Proof (for $\Theta = 0$)

$$p(x) = \int_{-\infty}^{\infty} m(x,y) \, dy$$

$$M(k_x,0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x,y) e^{-i2\pi(k_x x + k_y y)} \, dx \, dy =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x,y) e^{-i2\pi k_x x} \, dx \, dy =$$

$$\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} m(x,y) \, dy \right] e^{-i2\pi k_x x} \, dx =$$

$$= \int_{-\infty}^{\infty} p(x) e^{-i2\pi k_x x} \, dx = \mathcal{F}\{p(x)\}$$
Partly Discrete Reconstruction

- Let’s assume continuous angle $\Theta$, discrete $\rho$

\[ f(x, y) \]

\[ F(\kappa_x, \kappa_y) \]

\[ \kappa = \frac{\omega}{2\pi} \]

DTFT

\[ p_\theta[n] \Rightarrow P_\theta[\kappa] \]
Partly Discrete Reconstruction

• Let's assume continuous angle $\Theta$, discrete $\rho$

$$f(x, y)$$

$$y$$

$$x$$

$$F(\kappa_x, \kappa_y)$$

$$\kappa = \frac{\omega}{2\pi}$$

DTFT

$$p_\theta[n] \Rightarrow P_\theta[\kappa]$$
Reconstruction From Polar Coordinates

\[ f[n, m] = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} F(\kappa_x, \kappa_y) e^{2\pi j(\kappa_x n + \kappa_y m)} d\kappa_x d\kappa_y \]

\[ = \int_{0}^{\pi} \int_{-0.5}^{0.5} F(\rho, \theta) e^{2\pi j(\rho \cos(\theta)n + \rho \sin(\theta)m)} |\rho| d\rho d\theta \]

- Polar frequency data must be multiplied by \( |\rho| \)
- Also called a rho filter
Discrete Reconstruction

- Let’s assume discrete angle $\Theta_m$, discrete $\rho$

$$F(\kappa_x, \kappa_y)$$

$$\kappa = \frac{\omega}{2\pi}$$

$DFT$

$$p_{\theta_m}[n] \Rightarrow P_{\theta_m}[l]$$
Discrete Reconstruction

- Let's assume discrete angle $\Theta_m$, discrete $\rho$

\[
f(x, y)
\]

\[
F(\kappa_x, \kappa_y)
\]

\[
\kappa = \frac{\omega}{2\pi}
\]
Filtered Back Projection

• Replace integrals with sums. Sum over radius and angle

• Define a (filtered) backprojection:

\[ C_{\theta_m}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \theta_m] e^{2\pi j (l/N \cos(\theta_m) n_x + l/N \sin(\theta_m) n_y)} |l/N| \rho \]

So,

\[ f[n_x, n_y] = \sum_m C_{\theta_m}[n_x, n_y] \]
Example
Example Convolution Back Projection

- For $\Theta=0$

$$C_0[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, 0]|l/N|e^{2\pi j(l/N n_x)}$$
Example Convolution Back Projection

- For $\Theta = \pi/2$

\[
C_{\pi/2}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \pi/2]|l/N|e^{2\pi j (l/N n_y)}
\]
Convolution Back Projection
Filtered Back Projection

Back projection  Filtered Back projection
How Many Projections?

\[ F(\kappa_x, \kappa_y) \]

1/FOV

FOV
How Many Projections?

256 Proj.  128 Proj  64 Proj
Fan Beam CT

- Single Source
- Many detectors

- How to reconstruct?
Fan Beam CT

• Single Source
• Many detectors

• How to reconstruct?

• Re-binning!
Fan Beam CT

• Single Source
• Many detectors

• How to reconstruct?
• Re-binning!