Linear Filter Design

- Used to be an art
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- Both classes use finite order of parameters for design
- We will cover FIR designs, briefly mention IIR

What is a linear filter

- Attenuates certain frequencies
- Passes certain frequencies
- Effects both \textbf{phase} and \textbf{magnitude}
- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$, find an impulse response
  $$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

- Obtain the $M^{th}$ order causal FIR filter by truncating/windowing it
  $$h[n] = \begin{cases} 
  h_d[n]w[n] & 0 \leq n \leq M \\
  0 & \text{otherwise}
  \end{cases}$$

FIR Design by Windowing

- We already saw that, $H(e^{j\omega}) = H_d(e^{j\omega}) \ast W(e^{j\omega})$
- For Boxcar (rectangular) window
  $$W(e^{j\omega}) = e^{-j\omega \frac{M}{2}} \frac{\sin(w(M + 1)/2)}{\sin(w/2)}$$

FIR Design by Windowing

1. pass-band ripple
2. ideal
3. transition width
4. stop-band ripple
Tapered Windows

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<td>$\left[1 - \frac{\cos\left(\frac{\pi n}{M}\right)}{\cos\left(\frac{\pi}{2}\right)}\right]$</td>
<td>hamming(M+1)</td>
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Tradeoff - Ripple vs Transition Width

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- **FIR Filter Design**
  - Choose a desired frequency response $H_d(e^{j\omega})$
    - non causal (zero-delay), and infinite imp. response
    - If derived from C.T, choose $T$ and use:
      $$H_d(e^{j\omega}) = H_c(j\frac{\omega}{T})$$
  - **Window:**
    - Length $M+1 \leftrightarrow$ effect transition width
    - Type of window $\leftrightarrow$ transition-width/ripple
    - Modulate to shift impulse response
      $$H_d(e^{j\omega})e^{-j\omega \frac{M}{2}}$$

- **Example:** FIR Low-Pass Filter Design
  - $H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$
  - Choose $M \Rightarrow$ Window length and set
    $$H_1(e^{j\omega}) = H_d(e^{j\omega})e^{-j\omega \frac{M}{2}}$$
  - $h_1[n]$ and $H_1[n]$

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FIR Filter Design

- **FIR Filter Design**
  - Determine truncated impulse response $h_1[n]$
    $$h_1[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})e^{-j\omega\frac{M}{2}} e^{j\omega n} \, d\omega$$
    $$0 \leq n \leq M$$
    otherwise
  - Apply window
    $$h_w[n] = w[n]h_1[n]$$
  - **Check:**
    - Compute $H_w(e^{j\omega})$, if does not meet specs increase $M$ or change window

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Example: FIR Low-Pass Filter Design

- The result is a windowed sinc function
  $$h_w[n] = w[n]h_1[n]$$
  - **High Pass Design:**
    - Design low pass $h_0[n]$
    - Transform to $h_w[n] / |I|^s$
  - **General bandpass**
    - Transform to $2h_w[n] / \cos(\omega_0 n)$

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Characterization of Filter Shape

Time-Bandwidth Product, a unitless measure
\[ T(BW) = (M+1)\frac{\omega}{2\pi} \]  
also, total # of zero crossings

Larger TBW ⇒ More of the “sinc” function
hence, frequency response looks more like a rect function

Frequency Response Profile

Q: What are the lengths of these filters in samples?

\[ 2 = (M+1)\frac{(\pi/6)}{(2\pi)} \Rightarrow M=23 \]
\[ 12 = (M+1)\frac{(\pi)}{(2\pi)} \Rightarrow M=23 \]

Note that transition is the same!

Optimal Filter Design

• Window method
  – Design Filters heuristically using windowed sinc functions

• Optimal design
  – Design a filter \( h[n] \) with \( H(e^{j\omega}) \)
  – Approximate \( H_d(e^{i\omega}) \) with some optimality criteria - or satisfies specs.