Lecture 28
Phase Response
All-Pass and Minimum Phase

* Beautiful handwritten figures by Prof. Murat Arcak
GEOMAGNETIC STORM SPARKS AURORAS: Unsetted solar wind conditions + the possible arrival of a weak CME ignited a G2-class geomagnetic storm during the early hours of April 10th. Northern Lights spilled over the Canadian border into the USA as far south as Idaho, Montana and Colorado. The storm is subsiding now, but NOAA forecasters estimate a 50% chance that it could flare up again before the end of the day.
Phase response

Example:  \[ H(e^{j\omega}) = e^{j\omega n_d} \iff h[n] = \delta[n - n_d] \]

\[ |H(e^{j\omega})| = 1 \]

\[ \text{arg}[H(e^{j\omega})] = -\omega n_d \]

ARG is the wrapped phase
\text{arg} is the unwrapped phase
Group delay

To characterize general phase response, look at the group delay:

\[
grd[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}\]

For linear phase system, the group delay is \(n_d\)
Group delay

\[
grd[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}
\]

For narrowband signals, phase response looks like a linear phase
Group delay math

\[ H(z) = \frac{b_0}{a_0} \prod_{k=1}^{M} \frac{1 - c_k z^{-1}}{1 - d_k z^{-1}} \]

arg of products is sum of args

\[
\arg[H(e^{s w})] = - \sum_{k=1}^{N} \arg[1 - d_k e^{-s w}] \\
+ \sum_{k=1}^{M} \arg[1 - c_k e^{-s w}]
\]

\[
\text{grd}[H(e^{s w})] = - \sum_{k=1}^{N} \text{grd}[1 - d_k e^{-s w}] \\
+ \sum_{k=1}^{M} \text{grd}[1 - c_k e^{-s w}]
\]
Group delay math

\[ \text{grd} [H(e^{j\omega})] = -\sum_{k=1}^{N} \text{grd} [1 - de^{-j\omega}] + \sum_{k=1}^{M} \text{grd} [1 - ce^{-j\omega}] \]

Look at each factor:

\[ \text{arg} \left[ \frac{1 - re^{j\theta}e^{-j\omega}}{c_0 + de} \right] = \tan^{-1} \left( \frac{rsin(\omega - \theta)}{1 - re^{j\theta}(\omega - \theta)} \right) \]

\[ \text{grd} [1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - rcos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2} \]
Look at a zero lying on the real axis

Geometric Interpretation (for $\theta = 0$)

$$\arg [1 - re^{-i\omega}] = \arg [(e^{i\omega}r)e^{-i\omega}] = \arg [e^{i\omega}r] - \arg [e^{i\omega}]$$

$\phi$

$$\arg [1 - re^{-i\omega}]$$

$\theta \neq 0 \Rightarrow$ shift to the right by $\theta$
* Poles increase magnitude, but introduce phase lag and group delay.
* Zeros do the opposite.
* These effects are more marked when $r \to 1$. 

- **Log magnitude**: 
  - $r = 0.5$
  - $r = 0.7$
  - $r = 0.9$
  - $r = 1$

- **Phase**: 

- **Group delay**: 

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2nd order IIR example

Example: 2nd order IIR with complex poles

\[ H(z) = \frac{1}{(1-re^{i\theta}z^{-1})(1-re^{-i\theta}z^{-1})} \]

Log magnitude

Group delay

Phase
3nd order IIR example

Example: 3rd order IIR
All-Pass Systems

What is the magnitude response of

\[ H(z) = \frac{z^{-1} - a^*}{1 - a^* z^{-1}} \]

\[ |H(e^{j\omega})| = \left| \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \right| = \frac{|e^{-j\omega}(1 - a^* e^{j\omega})|}{|1 - ae^{-j\omega}|} = \frac{|1 - a^* e^{j\omega}|}{|1 - (a^* e^{j\omega})^*|} = 1 \quad \forall \omega \]
A general all-pass system:

\[ H_{ap}(z) = \prod_{k=1}^{M_{\text{Real}}} \frac{z^{-1}}{1 - d_k z^{-1}} \cdot \prod_{k=1}^{M_{\text{Complex}}} \frac{z^{-1} - e_k^*}{z^{-1} - e_k} \]

\( d_k \): real poles
\( e_k \): complex poles paired w/ conjugate \( e_k^* \)

\[ |H_{ap}(e^{j\omega})| = 1 \]

**Example**

[Diagram showing poles and zeros on the complex plane]
Phase response of an all-pass:

\[ \arg \left[ \frac{e^{-j\omega} - re^{j\Theta}}{1 - re^{j\Theta} e^{-j\omega}} \right] = \arg \left[ \frac{e^{-j\omega} (1 - re^{-j\Theta} e^{j\omega})}{1 - re^{j\Theta} e^{-j\omega}} \right] = \arg \left[ e^{-j\omega} \right] - 2\arg \left[ e^{-j\Theta} \right] \]

\[ \gcd \left[ \frac{e^{-j\omega} - re^{-j\Theta}}{1 - re^{j\Theta} e^{-j\omega}} \right] = 1 - 2\gcd \left[ e^{-j\Theta} \right] \]
Example:

Figure 5.20:

- Can be used to compensate phase distortion.
Claim: for a stable op system $H_{ap}(z)$:

(i) $\text{grd } [H_{ap}(e^{j\omega})] > 0$

(ii) $\text{arg } [H_{ap}(e^{j\omega})] \leq 0$

Delay positive $\rightarrow$ causal
Phase negative $\rightarrow$ phase lag.

Proof in book.