EE123
Digital Signal Processing

z-Transform
Today

• Last time:
  – DTFT - Ch 2

• Today:
  – finish DTFT
  – Z-Transform briefly!
  – Ch. 3

• Don’t forget -- ham lectures 6:30pm!
Somthing Fun

• goTenna
  – Text messaging radio
  – Bluetooth phone interface
  – MURS VHF radio (5channels)
  – 2W
  – 0.5-5 mile range
  – encryption
  – 2x100$

• Lab 6 implements a similar approach -- but without the slick system integration

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Frequency Response of LTI Systems

Check response to a pure frequency:

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right) e^{j\omega_0 n} \]

\[ H(e^{j\omega}) \bigg|_{\omega=\omega_0} \]
Frequency Response of LTI Systems

Check response to a pure frequency:

\[ e^{j\omega_0 n} \rightarrow \text{LTI} \rightarrow y[n] \]

\[ H(e^{j\omega}) = \text{DTFT}\{h[n]\} \]

\[ y[n] = H(e^{j\omega}) \bigg|_{\omega=\omega_0} e^{j\omega_0 n} \]

Output is the same pure frequency, scaled and phase-shifted!

\[ e^{j\omega_0 n} \] is an eigen function of LTI systems

Recall eigen vectors satisfy: \( A\nu = \lambda\nu \)
Example 3

Frequency response of a causal moving average filter

\[ y[n] = \frac{x[n - M] + \cdots + x[n]}{M + 1} \]

Q: What type of filter is it?  A: Low-Pass

\[ h[n] = \frac{1}{M + 1} w[n - \frac{M}{2}] \]
Example 3 Cont.

Frequency response of a causal moving average filter

\[ h[n] = \frac{1}{M + 1} w[n - \frac{M}{2}] \]

Same as example 1, only: Shifted by N, divided by M+1, M=2N

\[ H(e^{j\omega}) = \frac{e^{-j\omega \frac{M}{2}}}{M + 1} \cdot \frac{\sin \left( \left( \frac{M}{2} + \frac{1}{2} \right)\omega \right)}{\sin \left( \frac{\omega}{2} \right)} \]
Example 3 Cont.

Frequency response of a causal moving average filter

\[ H(e^{j\omega}) = \frac{e^{-j\omega \frac{M}{2}} \sin \left( \frac{\frac{M}{2} + 1}{2} \omega \right)}{M + 1} \cdot \frac{\sin \left( \frac{\omega}{2} \right)}{\sin \left( \frac{\omega}{2} \right)} \]

Not a sinc!
Example 4:

Impulse Response of an Ideal Low-Pass Filter

\[ h_{\text{LP}}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\text{LP}}(e^{j\omega})e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]
Example 4

Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi jn} e^{j\omega_c n} \bigg|_{-\omega_c}^{\omega_c} = 2j \sin(\omega_c n) \]

\[ = \frac{\sin(\omega_c n)}{\pi n} \]
Example 4

Impulse Response of an Ideal Low-Pass Filter

\[ h_{\text{LP}}[n] = \frac{\sin(\omega_c n)}{\pi n} \]

sampled “sinc”

Non causal! **Truncate** and **shift right** to make causal

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Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! **Truncate** and **shift right** to make causal

How does it change the frequency response?

Truncation:

\[
\tilde{h}_{LP}[n] = w_N[n] \cdot h_{LP}[n]
\]

property 2.9.7:

\[
\tilde{H}_{LP}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta
\]

Periodic convolution
Example 4

We get “smearing” of the frequency response
We get rippling
The z-Transform

• Used for:
  – Analysis of LTI systems
  – Solving difference equations
  – Determining system stability
  – Finding frequency response of stable systems
Eigen Functions of LTI Systems

- Consider an LTI system with impulse response $h[n]$: 

- We already showed that $x[n] = e^{j\omega n}$ are eigen-functions

- What if $x[n] = z^n = re^{j\omega n}$
Eigen Functions of LTI Systems

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \]

\[ = \left( \sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n = H(z)z^n \]

- \( x[n] = z^n \) are also eigen-functions of LTI Systems
- \( H(z) \) is called a transfer function
- \( H(z) \) exists for larger class of \( h[n] \) than \( H(e^{j\omega}) \)
The z Transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

· Since \( z = re^{j\omega} \)

\[ X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = DTFT\{x[n]\} \]
Region of Convergence (ROC)

• The ROC is a set of values of $z$ for which the sum

$$
\sum_{n=-\infty}^{\infty} x[n] z^{-n}
$$

Converges.
Region of Convergence (ROC)

- Example 1: Right-sided sequence $x[n] = a^n u[n]$

\[
X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n
\]

recall:

\[
1 + x + x^2 + \cdots = \frac{1}{1 - x}, \text{ if } |x| < 1
\]

So:

\[
X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC} = \{z : |z| > |a|\}
\]
Region of Convergence (ROC)

- Example 2: \( x[n] = (\frac{1}{2})^n u[n] + (-\frac{1}{3})^n u[n] \)

\[
X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}
\]

\[
\text{ROC} = \{ z : |z| > \frac{1}{2} \} \cap \{ z : |z| > \frac{1}{3} \} = \{ z : |z| > \frac{1}{2} \}
\]
Region of Convergence (ROC)

- Example 3: Left sided sequence  \( x[n] = -a^n u[-n - 1] \)

\[
X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{m=1}^{\infty} -a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1} z)^m
\]

if \( |a^{-1} z| < 1 \), i.e, \( |z| < |a| \) then,

\[
X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{-a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - az^{-1}}
\]
Region of Convergence (ROC)

• Expression is the same as Example 1!
• ROC = \{z : |z| < |a|\} is different

• The z-transform without ROC does not uniquely define a sequence!
Region of Convergence (ROC)

• Example 4: \( x[n] = -\left(\frac{1}{2}\right)^n u[-n - 1] + \left(-\frac{1}{3}\right)^n u[n] \)

\[
X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}
\]

\[
\text{ROC} = \{ z : |z| < \frac{1}{2} \} \cap \{ z : |z| > \frac{1}{3} \}
\]

\[
= \{ z : \frac{1}{3} < |z| < \frac{1}{2} \}
\]

Same as example 2
Region of Convergence (ROC)

- Example 5: \[ x[n] = \left( \frac{1}{2} \right)^n u[n] - \left( -\frac{1}{3} \right)^n u[-n - 1] \]

\[
\text{ROC} = \{ z : |z| > \frac{1}{2} \} \cap \{ z : |z| < \frac{1}{3} \} = 0
\]

- Example 6: \[ x[n] = a^n, \quad \text{two sided } a \neq 0 \]

\[
\text{ROC} = \{ z : |z| > a \} \cap \{ z : |z| < a \} = 0
\]
Region of Convergence (ROC)

- Example 7: Finite sequence \( x[n] = a^n u[n] u[-n + M - 1] \)

\[
X[z] = \sum_{n=0}^{M-1} a^n z^{-n} = \frac{1 - a^M z^{-M}}{1 - az^{-1}}
\]

Finite, always converges

Zero cancels pole

\( M-1 \)

\[
= \prod_{k=1}^{M-1} \left(1 - ae^{j \frac{2\pi k}{M}} z^{-1}\right)
\]

ROC \( = \{z : |z| > 0\} \)
Region of Convergence (ROC)
Properties of ROC

• A ring or a disk in Z-plane, centered at the origin

• DTFT converges iff ROC includes the unit circle

• ROC can’t contain poles
Properties of ROC

• For finite duration sequences, ROC is the entire z-plane, except possibly \( z=0, z=\infty \)

\[ X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0 \]

\[ X(z) = 1 + z^1 + z^2 \quad \text{ROC excludes } z = \infty \]
Properties of the ROC

- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  Examples 1, 2
- For left-sided: inwards from inner most pole to zero
  Example 3
- For two-sided, ROC is a ring - or do not exist
  Examples 4, 5, 6
Several Properties of the Z-transform

\[ x[n - n_d] \leftrightarrow z^{-n_d} X(z) \]

\[ z^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \]

\[ nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \]

\[ x[-n] \leftrightarrow X(z^{-1}) \]

\[ x[n] * y[n] \leftrightarrow X(z)Y(z) \]

ROC at least \( \text{ROC}_x \cap \text{ROC}_y \)
Inversion of the z-Transform

- In general, by contour integration within the ROC:
  \[ x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} \]

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem

- Most useful is the inverse of rational polynomials
  \[ X(z) = \frac{B(z)}{A(z)} \quad \text{Why?} \]