EE123

Digital Signal Processing

Lecture 5

based on slides by J.M. Kahn
Interesting Project: Pocket FM

http://www.pocket-fm.com
Info

- Last time
  - Finished DTFT Ch. 2
  - 12min z-Transforms Ch. 3
- Today: DFT Ch. 8
- Reminders:
  - HW Due tonight
  - Lab Checkoff next week
Motivation: Discrete Fourier Transform

• Sampled Representation in time and frequency
  – Numerical Fourier Analysis requires discrete representation
  – But, sampling in one domain corresponds to periodicity in the other...
  – What about DFS (DFT)?
    • Periodic in “time” ✓
    • Periodic in “Frequency” ✓
  – What about non-periodic signals?
    • Still use DFS(T), but need special considerations
Motivation: Discrete Fourier Transform

- Efficient Implementations exist
  - Direct evaluation of DFT: $O(N^2)$
  - Fast Fourier Transform (FFT): $O(N \log N)$
    (ch. 9, next topic....)
  - Efficient libraries exist: FFTW
    - In Python:
      > $X = \text{np.fft.fft}(x)$;
      > $x = \text{np.fft.ifft}(X)$;
  - Convolution can be implemented efficiently using FFT
    - Direct convolution: $O(N^2)$
    - FFT-based convolution: $O(N \log N)$
Discrete Fourier Series (DFS)

• Definition:
  – Consider N-periodic signal:
    \[ \tilde{x}[n + N] = \tilde{x}[n] \quad \forall n \]
    
    frequency-domain N-periodic representation:
    \[ \tilde{X}[k + N] = \tilde{X}[k] \quad \forall k \]
    
    – “~” indicates periodic signal/spectrum
Discrete Fourier Series (DFS)

- Define:
  \[ W_N \triangleq e^{-j2\pi/N} \]

- DFS:
  \[ \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \]
  \[ \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \]
Discrete Fourier Series (DFS)

- Properties of $W_N$:
  - $W_N^0 = W_N^N = W_N^{2N} = \ldots = 1$
  - $W_N^{k+r} = W_N^k W_N^r$ or $W_N^{k+N} = W_N^k$

- Example: $W_N^{kn}$ (N=6)
Discrete Fourier Transform

• By Convention, work with **one** period:

\[ x[n] \equiv \begin{cases} \tilde{x}[n] & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ X[k] \equiv \begin{cases} \tilde{X}[k] & 0 \leq k \leq N - 1 \\ 0 & \text{otherwise} \end{cases} \]

Same same..... but different!
Discrete Fourier Transform

• The DFT

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis} \]

\[ X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis} \]

• It is understood that,

\[ x[n] = 0 \quad \text{outside } 0 \leq n \leq N - 1 \]

\[ X[k] = 0 \quad \text{outside } 0 \leq k \leq N - 1 \]
Discrete Fourier Transform

- Alternative formulation (not in book)

Orthonormal DFT:

\[
x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{-kn}
\]

Inverse DFT, synthesis

\[
X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{kn}
\]

DFT, analysis

Why use this or the other?
Comparison between DFS/DFT

DFS

\[ \tilde{x}[n] \]

DFT

\[ x[n] \]
Example

- Take \( N=5 \)

\[
X[k] = \begin{cases} 
\sum_{n=0}^{4} W_5^{nk} & \text{if } k = 0, 1, 2, 3, 4 \\
0 & \text{otherwise}
\end{cases}
\]

\[
= 5\delta[k]
\]

“5-point DFT”
Example

• Q: What if we take N=10?

A: \[ X[k] = \tilde{X}[k] \] where \( \tilde{x}[n] \) is a period-10 seq.

\[ X[k] = \begin{cases} \sum_{n=0}^{4} W_{10}^{nk} & k = 0, 1, 2, \ldots, 9 \\ 0 & \text{otherwise} \end{cases} \]

“10-point DFT”
Example

- Show:

\[ X[k] = \sum_{n=0}^{4} W_{10}^{nk} \]

\[ = e^{-j \frac{4\pi}{10} k} \frac{\sin\left(\frac{\pi}{2} k\right)}{\sin\left(\frac{\pi}{10} k\right)} \]

“10-point DFT”
DFT vs DTFT

- For finite sequences of length N:
  - The N-point DFT of $x[n]$ is:
    $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \quad 0 \leq k \leq N - 1$$
  - The DTFT of $x[n]$ is:
    $$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \quad -\infty < \omega < \infty$$

What is similar?
DFT vs DTFT

• The DFT are samples of the DTFT at N equally spaced frequencies

\[ X[k] = X(e^{j\omega}) \bigg|_{\omega = k \frac{2\pi}{N}} \quad 0 \leq k \leq N - 1 \]
DFT vs DTFT

• Back to moving average example:

\[ X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} \]

\[ = e^{-j2\omega} \frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})} \]
FFTSHIFT

- Note that \( k=0 \) is \( w=0 \) frequency
- Use \texttt{fftshift} to shift the spectrum so \( w=0 \) in the middle.
DFT and Inverse DFT

• Both computed similarly.....let’s play:

\[
N \cdot x^*[n] = N \left( \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_{N}^{-kn} \right)^*
\]

\[
= \sum_{k=0}^{N-1} X^*[k] W_{N}^{kn}
\]

\[
= \mathcal{DFT} \{ X^*[k] \}.
\]

• Also....

\[
N \cdot x^*[n] = N \left( \mathcal{DFT}^{-1} \{ X[k] \} \right)^*.
\]
DFT and Inverse DFT

• So,

\[
\mathcal{DFT} \{X^*[k]\} = N \left( \mathcal{DFT}^{-1} \{X[k]\} \right)^*
\]

or,

\[
\mathcal{DFT}^{-1} \{X[k]\} = \frac{1}{N} \left( \mathcal{DFT} \{X^*[k]\} \right)^*
\]

• Implement IDFT by:
  – Take complex conjugate
  – Take DFT
  – Multiply by 1/N
  – Take complex conjugate

Why useful?
### DFT as Matrix Operator

#### DFT:
\[
\begin{pmatrix}
X[0] \\
\vdots \\
X[k] \\
\vdots \\
X[N-1]
\end{pmatrix}
= \begin{pmatrix}
W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)}
\end{pmatrix}
\begin{pmatrix}
x[0] \\
\vdots \\
x[n] \\
\vdots \\
x[N-1]
\end{pmatrix}
\]

#### IDFT:
\[
\begin{pmatrix}
x[0] \\
\vdots \\
x[n] \\
\vdots \\
x[N-1]
\end{pmatrix}
= \frac{1}{N}
\begin{pmatrix}
W_N^{-00} & \cdots & W_N^{-0k} & \cdots & W_N^{-0(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{-n0} & \cdots & W_N^{-nk} & \cdots & W_N^{-n(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{-(N-1)0} & \cdots & W_N^{-(N-1)k} & \cdots & W_N^{-(N-1)(N-1)}
\end{pmatrix}
\begin{pmatrix}
X[0] \\
\vdots \\
X[k] \\
\vdots \\
X[N-1]
\end{pmatrix}
\]

straightforward implementation requires $N^2$ complex multiplies :-(

M. Lustig, EECS UC Berkeley
DFT as Matrix Operator

• Can write compactly as:

\[ X = W_N x \]

\[ x = \frac{1}{N} W_N^* X \]

• So,

\[ x = \frac{1}{N} W_N^* X = \frac{1}{N} W_N^* W_N x = \frac{1}{N} (N I) x = x \]

as expected.

WHY?
Properties of DFT

• Inherited from DFS (EE120/20) so no need to be proved

• Linearity

\[ \alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k] \]

• Circular Time Shift

\[ x[((n - m))_N] \leftrightarrow X[k] e^{-j(2\pi/N)km} = X[k] W_N^{km} \]
Circular shift

\[ \tilde{x}[n] \]

\[ \tilde{x}[n-m] \]

\[ x[n] \]

\[ x[((n-m)\mod{N})] \]
Properties of DFT

• Circular frequency shift

\[ x[n]e^{j(2\pi/N)n^l} = x[n]W_N^{-nl} \leftrightarrow X[((k - l))_N] \]

• Complex Conjugation

\[ x^*[n] \leftrightarrow X^*[((-k))_N] \]

• Conjugate Symmetry for Real Signals

\[ x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N] \]

Show....
Examples

• 4-point DFT
  – Symmetry

• 5-point DFT
  – Symmetry
Examples

• 4-point DFT
  – Symmetry

• 5-point DFT
  – Symmetry
Properties of DFT

- Parseval’s Identity

\[
\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2
\]

- Proof (in matrix notation)

\[
x^*x = \left( \frac{1}{N} W_N^* X \right)^* \left( \frac{1}{N} W_N^* X \right) = \frac{1}{N^2} X^* \underbrace{W_N W_N^*}_N X = \frac{1}{N} X^* X
\]
Circular Convolution Sum

- **Circular Convolution:**

\[
x_1[n] \ast_N x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[(n - m)_N] = x_1[n] \ast_N x_2[n]
\]

for two signals of length N

- **Note:** Circular convolution is commutative

\[
x_2[n] \ast_N x_1[n] = x_1[n] \ast_N x_2[n]
\]
Properties of DFT

• Circular Convolution: Let $x_1[n]$, $x_2[n]$ be length $N$

$$x_1[n] \ast_N x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

Very useful!!! (for linear convolutions with DFT)

• Multiplication: Let $x_1[n]$, $x_2[n]$ be length $N$

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \ast_N X_2[k]$$
Linear Convolution

• Next....
  – Using DFT, circular convolution is easy
  – But, **linear** convolution is useful, not circular
  – So, show how to perform linear convolution with circular convolution
  – Used DFT to do linear convolution