Lecture 9
Spectral Analysis using DFT
Demo

• iSpectrum Demo
Announcements

• Last time:
  – FFT
• Today:
  – Frequency analysis with DFT
  – Windowing
  – Effect of zero-padding
Spectral analysis using the DFT

• DFT is a tool for spectrum analysis
• Should be simple:
  – Take a block, compute spectrum with DFT

• But, there are issues and tradeoffs:
  – Signal duration vs spectral resolution
  – Sampling rate vs spectral range
  – Spectral sampling rate
  – Spectral artifacts
Steps for processing C.T. signals:

Spectral analysis using the DFT
Spectral analysis using the DFT

- Two important tools:
  - Applying a window - reduced artifacts
  - Zero-padding - increases spectral sampling

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling interval</td>
<td>$T$</td>
<td>s</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>$\Omega_s = \frac{2\pi}{T}$</td>
<td>rad/s</td>
</tr>
<tr>
<td>Window length</td>
<td>$L$</td>
<td>unitless</td>
</tr>
<tr>
<td>Window duration</td>
<td>$L \cdot T$</td>
<td>s</td>
</tr>
<tr>
<td>DFT length</td>
<td>$N \geq L$</td>
<td>unitless</td>
</tr>
<tr>
<td>DFT duration</td>
<td>$N \cdot T$</td>
<td>s</td>
</tr>
<tr>
<td>Spectral resolution</td>
<td>$\Omega_s = \frac{2\pi}{L \cdot T}$</td>
<td>rad/s</td>
</tr>
<tr>
<td>Spectral sampling interval</td>
<td>$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$</td>
<td>rad/s</td>
</tr>
</tbody>
</table>
Filtered C.T Signal Example

\[ x_c(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \]

\[ X_c(j\Omega) = A_1 \pi [\delta(\Omega - \omega_1) + \delta(\Omega + \omega_1)] + A_2 \pi [\delta(\Omega - \omega_2) + \delta(\Omega + \omega_2)] \]
**Sampled Signal**

If we sampled the signal over an infinite time duration, we would have:

\[ x[n] = x_c(t)|_{t=nT}, \quad -\infty < n < \infty \]

described by the discrete-time Fourier transform:

\[
X(e^{j\Omega T}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c \left( j \left( \Omega - r \frac{2\pi}{T} \right) \right), \quad -\infty < \Omega < \infty
\]

Recall \( X(e^{j\omega}) = X(e^{j\Omega T}) \), where \( \omega = \Omega T \) ... more in ch 4.
In the examples shown here, the sampling rate is $\Omega_s/2\pi = 1/T = 20$ Hz, sufficiently high that aliasing does not occur.

Sampled Signal, $x[n] = x_c(nT)$, $-\infty < n < \infty$, $1/T = 20$ Hz

DTFT of Sampled Signal (heights represent areas of $\delta(\omega)$ impulses)
Block of $L$ Signal Samples

In any real system, we sample only over a finite block of $L$ samples:

$$x[n] = x_c(t)|_{t=nT}, \quad 0 \leq n \leq L - 1$$

This simply corresponds to a rectangular window of duration $L$.

Recall: in Homework 1 we explored the effect of rectangular and triangular windowing.
Windowed Block of $L$ Signal Samples
We take the block of signal samples and multiply by a window of duration $L$, obtaining:

$$v[n] = x[n] \cdot w[n], \quad 0 \leq n \leq L - 1$$

Suppose the window $w[n]$ has DTFT $W(e^{j\omega})$.

Then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega - \theta)})d\theta$$
Convolution with $W(e^{j\omega})$ has two effects in the spectrum:

1. It limits the spectral resolution. – Main lobes of the DTFT of the window
2. The window can produce *spectral leakage*. – Side lobes of the DTFT of the window

* These two are always a tradeoff - time-frequency uncertainty principle
## Windows (as defined in MATLAB)

<table>
<thead>
<tr>
<th>Name(s)</th>
<th>Definition</th>
<th>MATLAB Command</th>
<th>Graph ($M = 8$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular Boxcar</td>
<td>$w[n] = \begin{cases} 1 &amp;</td>
<td>n</td>
<td>\leq M/2 \ 0 &amp;</td>
</tr>
<tr>
<td>Triangular</td>
<td>$w[n] = \begin{cases} 1 - \frac{</td>
<td>n</td>
<td>}{M/2 + 1} &amp;</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$w[n] = \begin{cases} 1 - \frac{</td>
<td>n</td>
<td>}{M/2} &amp;</td>
</tr>
</tbody>
</table>

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# Windows (as defined in MATLAB)

<table>
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<th>Graph ((M = 8))</th>
</tr>
</thead>
</table>
| Hann    | \(w[n] = \begin{cases} 
\frac{1}{2} \left[ 1 + \cos \left( \frac{\pi n}{M/2} \right) \right] & |n| \leq M/2 \\
0 & |n| > M/2 
\end{cases}\) | hann(M+1) | ![Hann Window Graph](image1) |
| Hanning | \(w[n] = \begin{cases} 
\frac{1}{2} \left[ 1 + \cos \left( \frac{\pi n}{M/2+1} \right) \right] & |n| \leq M/2 \\
0 & |n| > M/2 
\end{cases}\) | hanning(M+1) | ![Hanning Window Graph](image2) |
| Hamming | \(w[n] = \begin{cases} 
0.54 + 0.46 \cos \left( \frac{\pi n}{M/2} \right) & |n| \leq M/2 \\
0 & |n| > M/2 
\end{cases}\) | hamming(M+1) | ![Hamming Window Graph](image3) |
Windows

- All of the window functions \( w[n] \) are real and even.
- All of the discrete-time Fourier transforms

\[
W(e^{j\omega}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n] e^{-jn\omega}
\]

are real, even, and periodic in \( \omega \) with period \( 2\pi \).

- In the following plots, we have normalized the windows to unit d.c. gain:

\[
W(e^{j0}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n] = 1
\]

This makes it easier to compare windows.
Window Example

$M = 16$

- Boxcar
- Triangular

$M = 16$

- Hanning
- Hamming

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Windows Properties

These are characteristic of the window type

<table>
<thead>
<tr>
<th>Window</th>
<th>Main-lobe</th>
<th>Sidelobe $\delta_s$</th>
<th>Sidelobe $-20 \log_{10} \delta_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rect</td>
<td>$\frac{4\pi}{M+1}$</td>
<td>0.09</td>
<td>21</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$\frac{8\pi}{M+1}$</td>
<td>0.05</td>
<td>26</td>
</tr>
<tr>
<td>Hann</td>
<td>$\frac{8\pi}{M+1}$</td>
<td>0.0063</td>
<td>44</td>
</tr>
<tr>
<td>Hamming</td>
<td>$\frac{12\pi}{M+1}$</td>
<td>0.0022</td>
<td>53</td>
</tr>
<tr>
<td>Blackman</td>
<td>$\frac{M+1}{M+1}$</td>
<td>0.0002</td>
<td>74</td>
</tr>
</tbody>
</table>

Most of these (Bartlett, Hann, Hamming) have a transition width that is twice that of the rect window.

**Warning:** Always check what’s the definition of M

Adapted from *A Course In Digital Signal Processing* by Boaz Porat, Wiley, 1997
Windows Examples

Here we consider several examples. As before, the sampling rate is \( \Omega_s/2\pi = 1/T = 20 \text{ Hz.} \)

**Rectangular Window, \( L = 32 \)**
Windows Examples

Triangular Window, $L = 32$

![Graph of Triangular Window, $L = 32$](image1)

DTFT of Triangular Window

![Graph of DTFT of Triangular Window](image2)

Sampled, Windowed Signal, Triangular Window, $L = 32$

![Graph of Sampled, Windowed Signal](image3)

DTFT of Sampled, Windowed Signal

![Graph of DTFT of Sampled, Windowed Signal](image4)
Windows Examples

Hamming Window, $L = 32$

- **Hamming Window, $L = 32$**
  - Plot of $w[n] = 0.54 - 0.46 \cos \left( \frac{2\pi n}{L} \right)$ with $n = 0, 1, \ldots, L-1$.

- **DTFT of Hamming Window**
  - Frequency response of the Hamming window.

- **Sampled, Windowed Signal, Hamming Window, $L = 32$**
  - Plot of the sampled and windowed signal $x[n] * w[n]$.

- **DTFT of Sampled, Windowed Signal**
  - Frequency response of the sampled and windowed signal.

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Windows Examples

Hamming Window, $L = 64$

Hamming Window, $L = 64$

Sampled, Windowed Signal, Hamming Window, $L = 64$

DTFT of Sampled, Windowed Signal

DTFT of Hamming Window
Optimal Window: Kaiser

• Minimum main-lobe width for a given side-lobe energy %

\[
\frac{\int_{\text{sidelobes}} |H(e^{j\omega})|^2 d\omega}{\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega}
\]

• Window is parametrized with L and \( \beta \)  
  – \( \beta \) determines side-lobe level  
  – L determines main-lobe width

OS Eq 10.12
Example - Noiseless

\[ y = \sin(2\pi 0.1992n) + 0.005 \sin(2\pi 0.25n) \quad | \quad 0 \leq n < 128 \]
In preparation for taking an $N$-point DFT, we may zero-pad the windowed block of signal samples to a block length $N \geq L$:

$$v[n] \quad 0 \leq n \leq L - 1$$

$$0 \quad L \leq n \leq N - 1$$

This zero-padding has no effect on the DTFT of $v[n]$, since the DTFT is computed by summing over $-\infty < n < \infty$.

**Effect of Zero Padding**

We take the $N$-point DFT of the zero-padded $v[n]$, to obtain the block of $N$ spectral samples:

$$V[k], \quad 0 \leq k \leq N - 1$$
Zero-Padding

Consider the DTFT of the zero-padded $v[n]$. Since the zero-padded $v[n]$ is of length $N$, its DTFT can be written:

$$V(e^{j\omega}) = \sum_{n=0}^{N-1} v[n]e^{-jn\omega}, \quad -\infty < \omega < \infty$$

The $N$-point DFT of $v[n]$ is given by:

$$V[k] = \sum_{n=0}^{N-1} v[n]W_N^{kn} = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)nk}, \quad 0 \leq k \leq N - 1$$

We see that $V[k]$ corresponds to the samples of $V(e^{j\omega})$:

$$V[k] = V(e^{j\omega})\bigg|_{\omega=k\frac{2\pi}{N}}, \quad 0 \leq k \leq N - 1$$

To obtain samples at more closely spaced frequencies, we zero-pad $v[n]$ to longer block length $N$. The spectrum is the same, we just have more samples.
Note that the ordering of the DFT samples is unusual.

$$V[k] = \sum_{n=0}^{N-1} v[n] W_N^{nk}$$

The DC sample of the DFT is $k = 0$

$$V[0] = \sum_{n=0}^{N-1} v[n] W_N^{0n} = \sum_{n=0}^{N-1} v[n]$$

- The positive frequencies are the first $N/2$ samples
- The first $N/2$ negative frequencies are circularly shifted

$$(−k)_N = N − k$$

so they are the last $N/2$ samples. (Use `fftshift` to reorder)
Frequency Analysis with DFT Examples:

Hamming Window, $L = 32$, $N = 32$

Sampled, Windowed Signal, Hamming Window, $L = 32$, Zero-Padded to $N = 32$

$N$-Point DFT of Sampled, Windowed, Zero-Padded Signal

Spectrum of Sampled, Windowed, Zero-Padded Signal

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Frequency Analysis with DFT Examples:

**Hamming Window, \( L = 32 \), Zero-Padded to \( N = 64 \)**

- **Sampled, Windowed Signal, Hamming Window, \( L = 32 \), Zero-Padded to \( N = 64 \)**
- **N-Point DFT of Sampled, Windowed, Zero-Padded Signal**
- **Spectrum of Sampled, Windowed, Zero-Padded Signal**

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A 40 yo pt with a history of lower limb weakness referred for MRI screening of brain and whole spine for cord. MRI sagittal T2 screening of dorsal region shows a faint uniform linear high signal at the center of the cord. The signal abnormality likely to represent:

(1) Cord demyelination.
(2) Syrinx (spinal cord disease).
(3) Artifact.

**Answer**: Its an artifact, known as truncation or Gibbs artifact.
Length of window determines spectral resolution

Type of window determines side-lobe amplitude.
(Some windows have better tradeoff between resolution-sidelobe)

Zero-padding approximates the DTFT better. Does not introduce new information!
<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Spectral error from aliasing</td>
<td>a. Filter signal to reduce frequency content above $\Omega_s/2 = \pi/T$.</td>
</tr>
<tr>
<td></td>
<td>b. Increase sampling frequency $\Omega_s = 2\pi/T$.</td>
</tr>
<tr>
<td>2. Insufficient frequency</td>
<td>a. Increase $L$</td>
</tr>
<tr>
<td>resolution.</td>
<td>b. Use window having narrow main lobe.</td>
</tr>
<tr>
<td></td>
<td>b. Increase $L$</td>
</tr>
<tr>
<td>4. Missing features due to</td>
<td>a. Increase $L$,</td>
</tr>
<tr>
<td>spectral sampling.</td>
<td>b. Increase $N$ by zero-padding $v[n]$ to length $N &gt; L$.</td>
</tr>
</tbody>
</table>