Lecture 11
Introduction to Wavelets
Discrete STFT

\[ X[r, k] = \sum_{m=0}^{L-1} x[rR + m]w[m]e^{-j2\pi km/N} \]

\[ \Delta \omega = \frac{2\pi}{L} \]

\[ \Delta t = L \]

one STFT coefficient
Limitations of Discrete STFT

• Need overlapping ⇒ Not orthogonal

• Computationally intensive $O(MN \log N)$

• Same size Heisenberg boxes
From STFT to Wavelets

• Basic Idea:
  – low-freq changes slowly - fast tracking unimportant
  – Fast tracking of high-freq is important in many apps.
  – Must adapt Heisenberg box to frequency

• Back to continuous time for a bit.....
From STFT to Wavelets

- Continuous time

\[ S f(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t - u)e^{-j\Omega t} dt \]

\[ W f(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*\left(\frac{t - u}{s}\right) dt \]

*Morlet - Grossmann*
From STFT to Wavelets

\[ W_f(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t-u}{s}) dt \]

- The function \( \Psi \) is called a mother wavelet
  - Must satisfy:
    \[
    \int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \Rightarrow \text{unit norm}
    \]
    \[
    \int_{-\infty}^{\infty} \Psi(t) dt = 0 \Rightarrow \text{Band-Pass}
    \]
STFT and Wavelets “Atoms”

**STFT Atoms**
(with hamming window)

\[ w(t - u)e^{j\Omega t} \]

\[ \Omega_{hi} \]

\[ \Omega_{lo} \]

**Wavelet Atoms**

\[ \frac{1}{\sqrt{s}} \Psi\left(\frac{t - u}{s}\right) \]

s = 1

s = 3
Examples of Wavelets

- **Mexican Hat**
  \[ \Psi(t) = (1 - t^2)e^{-t^2/2} \]

- **Haar**
  \[ \Psi(t) = \begin{cases} 
  -1 & 0 \leq t < \frac{1}{2} \\
  1 & \frac{1}{2} \leq t < 1 \\
  0 & \text{otherwise} 
\end{cases} \]
Example: Wavelet of Chirp
Wavelets VS STFT
Example 2: “Bumpy” Signal

\[ \log(s) \]

Sombrero Wavelet

\[ \log(s) \]

\[ u \]
Wavelets Transform

• Can be written as linear filtering

\[ Wf(u, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t) \Psi^*(\frac{t-u}{s}) dt \]

\[ = \left\{ f(t) * \overline{\Psi}_s(t) \right\}(u) \]

\[ \overline{\Psi}_s = \frac{1}{\sqrt{s}} \Psi(\frac{t}{s}) \]

• Wavelet coefficients are a result of bandpass filtering
Wavelet Transform

- Many different constructions for different signals
  - Haar good for piece-wise constant signals
  - Battle-Lemarie’: Spline polynomials

- Can construct Orthogonal wavelets
  - For example: dyadic Haar is orthonormal

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right)$$

$$i = [1, 2, 3, \cdots]$$
Orthonormal Haar

Same scale non-overlapping

Orthogonal between scales
Scaling function

\[ \Psi_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right) \]

• Problem:
  – Every stretch only covers half remaining bandwidth
  – Need Infinite functions

recall, for chirp:
Scaling function

$$\Psi_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right)$$

- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions

- Solution:
  - Plug low-pass spectrum with a scaling function $$\Phi$$
Haar Scaling function

\[ \Psi(t) = \begin{cases} 
-1 & 0 \leq t < \frac{1}{2} \\
1 & \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise} 
\end{cases} \]

\[ \Phi(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
0 & \text{otherwise} 
\end{cases} \]
Back to Discrete

• Early 80’s, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
• Late 80’s link to DSP by Daubechies and Mallat.

• From CWT to DWT not so trivial!
• Must take care to maintain properties