Lecture 12
Discrete Wavelet Transform
Announcements

• Midterm I
  – This coming Monday. 10am-12pm in class
  – Everything till today (including)
  – Open everything -- except electronics

• Pre-Lab II Due Thursday.
  – Install SDR drivers
  – Install SDR software
  – Look at different parts of the spectrum
Back to Discrete

• Early 80’s, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
• Late 80’s link to DSP by Daubechies and Mallat.

• From CWT to DWT not so trivial!
• Must take care to maintain properties
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]

\[
\begin{array}{cccc}
  & d_{00} & d_{01} & d_{02} & d_{03} \\
 d_{10} & & & & \\
a_{10} & & a_{11} & & \\
\end{array}
\]
Example: Discrete Haar Wavelet

Haar for n=2

Equivalent to DFT_2!
Discrete Orthogonal Haar Wavelet

Haar for n=8

scaling

$\Phi_{20}$

$\Psi_{20}$

$\Psi_{10}$

$\Psi_{11}$

$\Psi_{00}$

$\Psi_{01}$

$\Psi_{02}$

$\Psi_{03}$
Discrete Orthogonal Haar Wavelet

\( \Psi_{00} \)

\( \Psi_{01} \)

\( \Psi_{02} \)

\( \Psi_{03} \)

\( |\mathcal{F}\{\Psi_{0x}(e^{j\omega})\}| \)

0

\( \pi \)

\( t \)

\( \omega \)
Discrete Orthogonal Haar Wavelet

$|\mathcal{F}\{\Psi_{1x}(e^{j\omega})\}|$

$\Pi$

$0$

$\psi_{10}$

$\psi_{11}$

$\frac{1}{2}$

$\frac{1}{2}$

$\gamma$

M. Lustig, EECS UC Berkeley
Discrete Orthogonal Haar Wavelet

Scaling

$\Phi_{20}$

$\Psi_{20}$

$|\mathcal{F}\{\Phi_{2x}(e^{j\omega})\}|$  $|\mathcal{F}\{\Psi_{2x}(e^{j\omega})\}|$

$0$  $\pi$

$\omega$

$t$
Optional: stop decomposition at Level 1

scaling

\[ \Phi_{10} \]

\[ \Phi_{11} \]

\[ |\mathcal{F}\{\Phi_{1x}(e^{j\omega})\}| \]
Fast DWT with Filter Banks (more Later!)

\[ x[n] \xrightarrow{h_0[n]} a_{0n} \]
\[ \cdots \]
\[ h_1[n] \xrightarrow{d_{0n}} \]

not quite... too many coefficients
Fast DWT with Filter Banks

\[ x[n] \rightarrow h_0[n] \rightarrow h_1[n] \rightarrow a_{0n} \rightarrow d_{0n} \]
Fast DWT with Filter Banks

\[ h_0[n] \rightarrow h_1[n] \]

\[ x[n] \]

Complexity: \[ N + N/2 + N/4 + N/8 + \ldots + = 2N = O(N) \]
Decomposition

\[ x[n] \rightarrow h_0[n] \rightarrow \downarrow 2 \rightarrow a_{0n} \]

\[ h_0[n] \rightarrow \downarrow 2 \rightarrow \bullet \rightarrow d_{0n} \]

\[ h_1[n] \rightarrow \downarrow 2 \rightarrow \bullet \]

\[ h_1[n] \rightarrow \downarrow 2 \rightarrow d_{1n} \]
Reconstruction

Just flip arrows, replace $h$ with $g$. This diagram demonstrates the process of reconstruction in signal processing, where inputs $h_0[n]$ and $h_1[n]$ are transformed to outputs $g_0[n]$ and $g_1[n]$, with $x[n]$ being the input signal to the reconstruction process.
Example, Haar DWT - Level 0

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example_haar_dwt_level_0.png}
\caption{Example of Haar Discrete Wavelet Transform (DWT) - Level 0}
\end{figure}
Example, Haar DWT - Level 1
Example, Haar DWT - Level 2
Example, Haar DWT - Level 3
Example, Haar DWT - Level 4
Example, Haar DWT - Level 5
DWT Another view

[Image of a graph and a histogram]
Haar DWT Example

$x[n]$
Approximation from 25/256 coefficients

**Haar**

**DFT**
Example: Denoising Noisy Signals

Haar
Example: Denoising by Thresholding

noisy

largest 25 coefficients

denoised
Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet  Jpeg - DCT

@ 66 fold compression ratio
Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet  Jpeg - DCT

@ 66 fold compression ratio
Approximation/Compression

0.000 % coefficients
Example in Research

Robust 4D Flow Denoising using Divergence-free Wavelet Transform

Frank Ong\textsuperscript{1}, Martin Uecker\textsuperscript{1}, Umar Tariq\textsuperscript{2}, Albert Hsiao\textsuperscript{2}, Marcus T Alley\textsuperscript{2}, Shreyas S Vasanawala\textsuperscript{2}, Michael Lustig\textsuperscript{1}

courtesy, Frank Ong and Marcus Alley
Noisy Flow Data

Vector visualization
Streamline visualization

Emitter plane from descending aorta
Analysis plane from ascending aorta

Original
DFW Manual Threshold
Divergence Free Wavelets

(a) Linear spline $\Phi_0$
   Quadratic spline $\Phi_1$
   Linear spline $\psi_0$
   Quadratic spline $\psi_1$

(b) Illustration of wavelet decomposition

(c) Illustration of wavelet reconstruction

FWT (vx) FWT (vy) FWT (vz)
IWT (vx) IWT (vy) IWT (vz)
Divergence-Free Wavelet Denoising

Vector visualization

Streamline visualization

Emitter plane from descending aorta

Analysis plane

Emitter plane from ascending aorta

Original DFW Manual Threshold
Noisy Flow Data

- DivFree & Non-DivFree Thresh
- Non-DivFree Thresh Only

Original
- DivFree Wavelet Manual Threshold

Vector visualization
Streamline visualization

Emitter plane from descending aorta
Analysis plane from ascending aorta
Emitter plane from ascending aorta