Lecture 18
Filter Banks
Last Time

• Exchange of filtering and expanders

• Today:
  – Exchange of filtering and compressors
  – Polyphase decomposition
  – Multi-rate Filter Banks
  – Subtleties in Time-Frequency tiling
  – Perfect reconstruction with non-ideal filters
  – Polyphase filter banks
Polyphase Decomposition

- We can decompose an impulse response to:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]
Polyphase Decomposition

• Define:

\[ h_k[n] \rightarrow \downarrow M \rightarrow e_k[n] \]

\[ e_k[n] = h_k[nM] \]
Polyphase Decomposition

\[ e_k[n] \rightarrow \uparrow M \rightarrow h_k[n] \]

recall upsampling \(\Rightarrow\) scaling

\[ H_k(z) = E_k(z^M) \]

Also, recall:

\[ h[n] = \sum_{k=0}^{M-1} h_k[n - k] \]

So,

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \]
Polyphase Decomposition

\[ H(z) = \sum_{k=0}^{M-1} E_k(z^M)z^{-k} \]

Why should you care?
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]

- **Problem:**
  - Compute all \( y[n] \) and then throw away -- wasted computation!
    - For FIR length \( N \) \( \Rightarrow N \) mults/unit time
  - Can interchange Filter with compressor?
    - Not in general!
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]
Polyphase Implementation of Decimation

\[ x[n] \rightarrow H(z) \rightarrow y[n] \rightarrow \downarrow M \rightarrow w[n] = y[nM] \]

Interchange filter with decimation

\[ x[n] \rightarrow \begin{array}{c} E_0(z^M) \rightarrow \downarrow M \\ z^{-1} \rightarrow E_1(z^M) \rightarrow \downarrow M \\ z^{-1} \rightarrow \vdots \\ z^{-1} \rightarrow E_{M-1}(z^M) \rightarrow \downarrow M \end{array} \rightarrow + \rightarrow w[n] \]

now, what can we do?
Polyphase Implementation of Decimation

\[ x[n] \xrightarrow{H(z)} y[n] \xrightarrow{\downarrow M} w[n] = y[nM] \]

Interchange filter with decimation

\[ x[n] \xrightarrow{\downarrow M} z^{-1} \xrightarrow{\downarrow M} E_0(z) \xrightarrow{\downarrow M} E_1(z) \xrightarrow{\downarrow M} \cdots \xrightarrow{\downarrow M} E_{M-1}(z) \xrightarrow{+} w[n] \]

Computation:
Each Filter: \( N/M \times (1/M) \) mult/unit time
Total: \( N/M \) mult/unit time

what about interpolation?
Multirate FilterBank

- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
- Often $h_1[n] = e^{j\pi n} h_0[n]$ or $H_1(e^{j\omega}) = H_0(e^{j(w-\pi)})$
Subtleties in Time-Freq Tiling

- Assume \( h_0, h_1 \) are ideal low, high pass filters
Subtleties in Time-Freq Tiling

- Assume $h_0$, $h_1$ are ideal low, high pass filters

\[ x[n] \xrightarrow{h_0[n]} h_1[n] \xrightarrow{\downarrow 2} \]
Subtleties in Time-Freq Tiling

- Assume \( h_0, h_1 \) are ideal low, high pass filters
Subtleties in Time-Freq Tiling

• Assume $h_0, h_1$ are ideal low, high pass filters
Perfect Reconstruction Ideal Filters

\[ \begin{align*}
& g_0[n] \\
& g_1[n] \\
\end{align*} \]

\[ y[n] \]

\[ \begin{align*}
2 & \uparrow \\
2 & \uparrow \\
\end{align*} \]

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Non ideal LP and HP Filters

\[ x[n] \xrightarrow{\text{h}_0[n]} \downarrow 2 \xrightarrow{\text{h}_1[n]} \downarrow 2 \]

\[ X(e^{j\omega}) \quad H_0(e^{j\omega}) \quad H_1(e^{j\omega}) \]
Perfect Reconstruction non-Ideal Filters

\[ x[n] \rightarrow h_0[n] \downarrow 2 \rightarrow \text{Stuff} \rightarrow 2 \uparrow \rightarrow g_0[n] \rightarrow y[n] \]

\[ x[n] \rightarrow h_1[n] \downarrow 2 \rightarrow \text{Stuff} \rightarrow 2 \uparrow \rightarrow g_1[n] \rightarrow y[n] \]

\[ X(e^{j\omega}) \rightarrow \cdot G_0(e^{j\omega}) \rightarrow + \rightarrow \cdot G_1(e^{j\omega}) \rightarrow = X(e^{j\omega}) \]
Perfect Reconstruction non-Ideal Filters

\[ Y(e^{j\omega}) = \frac{1}{2} \left[ G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega}) \right] X(e^{j\omega}) \]

\[ + \frac{1}{2} \left[ G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)}) \right] X(e^{j(\omega-\pi)}) \]

need to cancel!
Quadrature Mirror Filters - perfect recon

$$H_1(e^{j \omega}) = H_0(e^{j(\omega - \pi)})$$

$$G_0(e^{j \omega}) = 2H_0(e^{j \omega})$$

$$G_1(e^{j \omega}) = -2H_1(e^{j \omega})$$
Quadrature Mirror Filters - perfect recon

\[
H_1(e^{j\omega}) = H_0(e^{j(\omega - \pi)}) \\
G_0(e^{j\omega}) = 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) = -2H_1(e^{j\omega})
\]

Example Haar:

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\[ e_{00} = h_0[2n] \]
\[ e_{01} = h_0[2n + 1] \]
\[ e_{10} = h_1[2n] = e^{j2\pi n} h_0[2n] = e_{00}[n] \]
\[ e_{11} = h_1[2n + 1] = e^{j2\pi n} e^{j\pi} h_0[2n + 1] = -e_{01}[n] \]
Polyphase Filter-Bank

\[ x[n] \]

\[ z^{-1} \]

\[ \downarrow 2 \]

\[ e_{00}[n] \]

\[ + \]

\[ z^{-1} \]

\[ \downarrow 2 \]

\[ e_{01}[n] \]

\[ + \]

\[ \downarrow 2 \]

\[ e_{10}[n] \]

\[ + \]

\[ \downarrow 2 \]

\[ e_{11}[n] \]

\[ e_{00} = h_0[2n] \]

\[ e_{01} = h_0[2n+1] \]

\[ e_{10} = e_{00}[n] \]

\[ e_{11} = -e_{01}[n] \]
Polyphase Filter-Bank

\[ x[n] \]

\[ z^{-1} \]

\[ e_{00}[n] \]

\[ e_{01}[n] \]

\[ e_{10}[n] \]

\[ e_{11}[n] \]

\[ e_{00} = h_0[2n] \]

\[ e_{01} = h_0[2n + 1] \]

\[ e_{10} = e_{00}[n] \]

\[ e_{11} = -e_{01}[n] \]