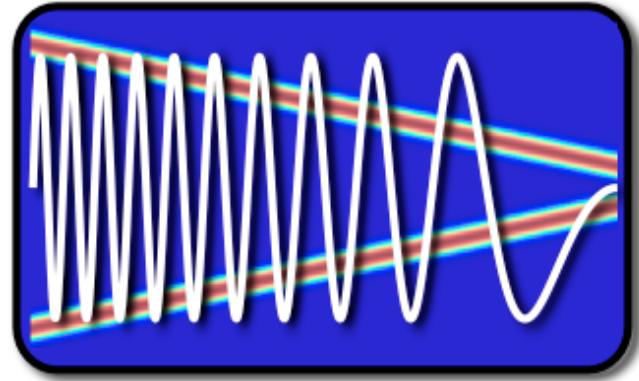


EE123



# Digital Signal Processing

## Lecture 22 Transform Analysis of LTI Systems

# IIR Design

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- Historically
  - Continuous IIR design was advanced
  - Use results from C.T to D.T
  - C.T IIR designs have closed form, easy to use
  - Easy to control Magnitude, not easy to control phase
- Common Types:
  - Butterworth - monotonic, no ripple
  - Chebyshev - Type I, pass band ripple, Type II stop band ripple
  - Elliptic - Ripples in both bands

## Design of D.T IIR Filters from Analog

- Discretize by one of many techniques
- $H_c(s) \Rightarrow H(z)$
- Must satisfy:
  - Imaginary axis is mapped to unit circle
  - stability of  $H_c(s)$  should result in stable  $H(z)$
- Two methods:
  - Impulse invariance - match impulse response
  - Bilinear transformation

`scipy.signal.iirdesign`

## Impulse Invariance

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- Match the impulse response

$$h[n] = Th_c(nT)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c(j\Omega + j\Omega_s k) \Big|_{\Omega=\frac{\omega}{T}}$$

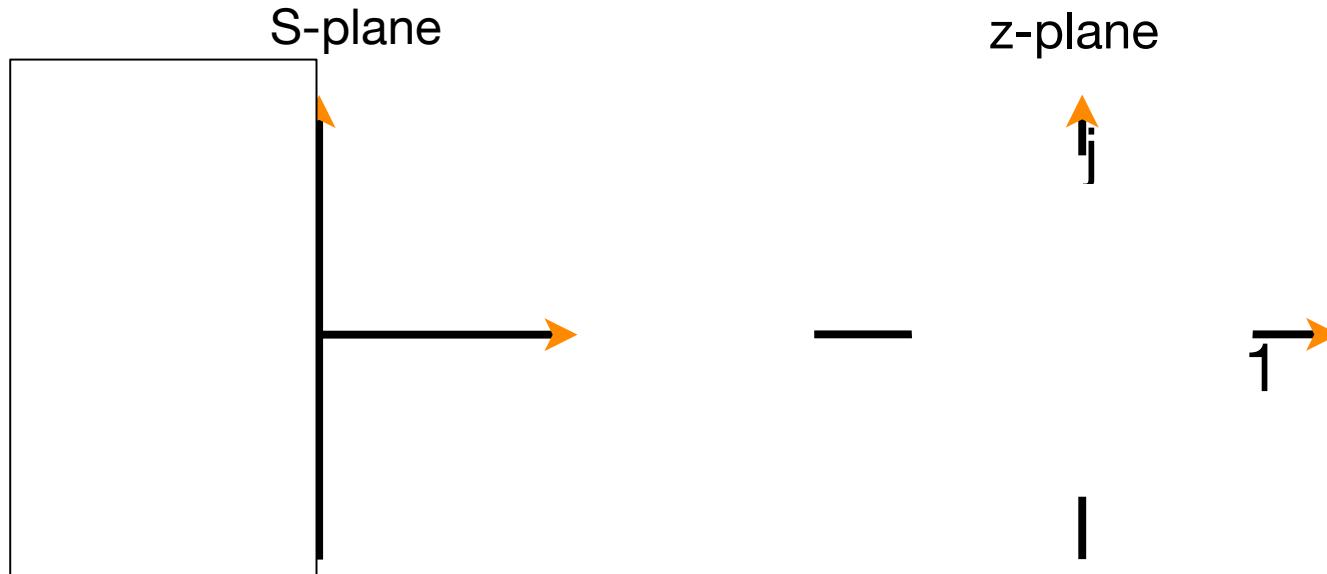
- If  $H_c(j\Omega)$  is not bandlimited, then the frequency response will alias.

## Bilinear Transformation

- Warp  $H_c(j\Omega)$  such that it is bandlimited

$$S = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$z = \frac{1 + \frac{T}{2}S}{1 - \frac{T}{2}S}$$



- Need to predistort continuous parameters to get the right discrete freq. response

# Rational system response

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Linear difference equations :

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Example:  $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

## Direct Design of IIR Filters

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- Idea: Optimize directly - not easy!

$$\operatorname{argmin} \int_{\omega \in \text{care}} \left| \frac{B(e^{j\omega})}{A(e^{j\omega})} - H_d(e^{j\omega}) \right|^2 d\omega$$

- Approximate using weighted Linear Least squares

$$\operatorname{argmin} \int_{\omega \in \text{care}} W(\omega)^2 |B(e^{j\omega}) - H_d(e^{j\omega})A(e^{j\omega})|^2 d\omega$$

- Discretize and solve... but....
  - Issues with guaranteeing stability
  - Hard to prescribe feasible desired complex response

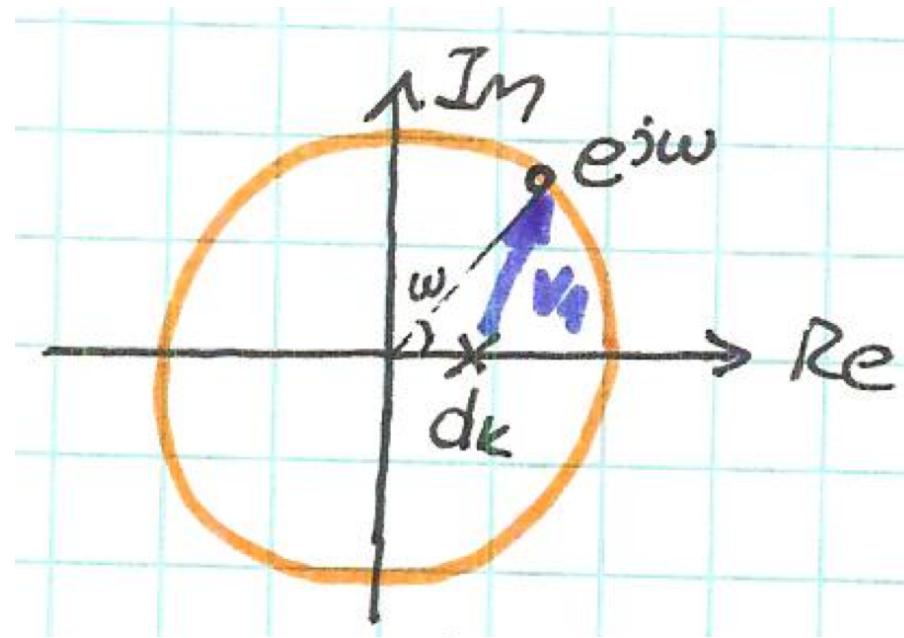
# Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

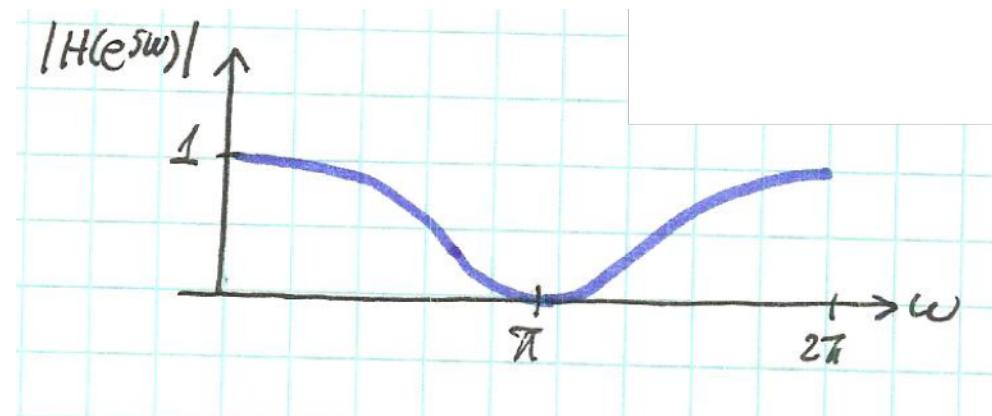
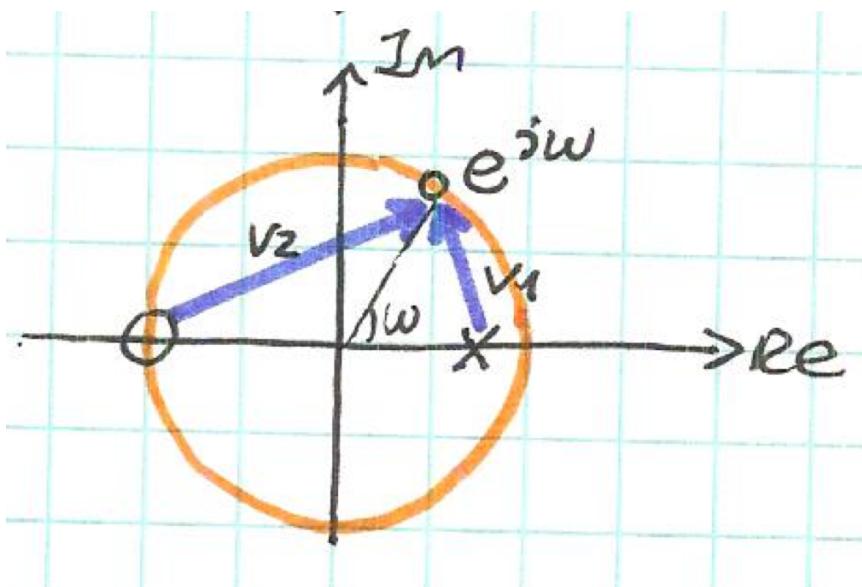


# Magnitude Response Example

Example:

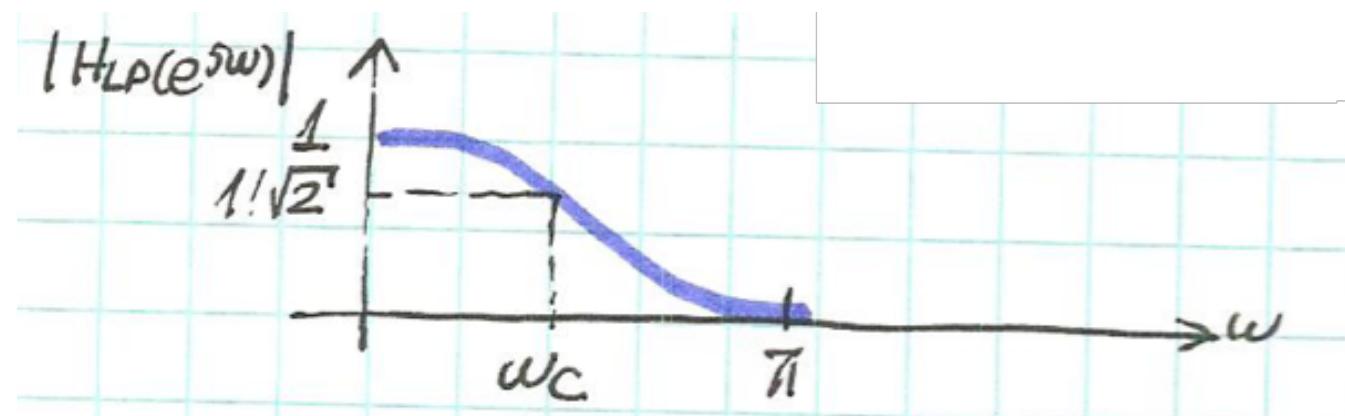
$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



## Simple low pass filter

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$

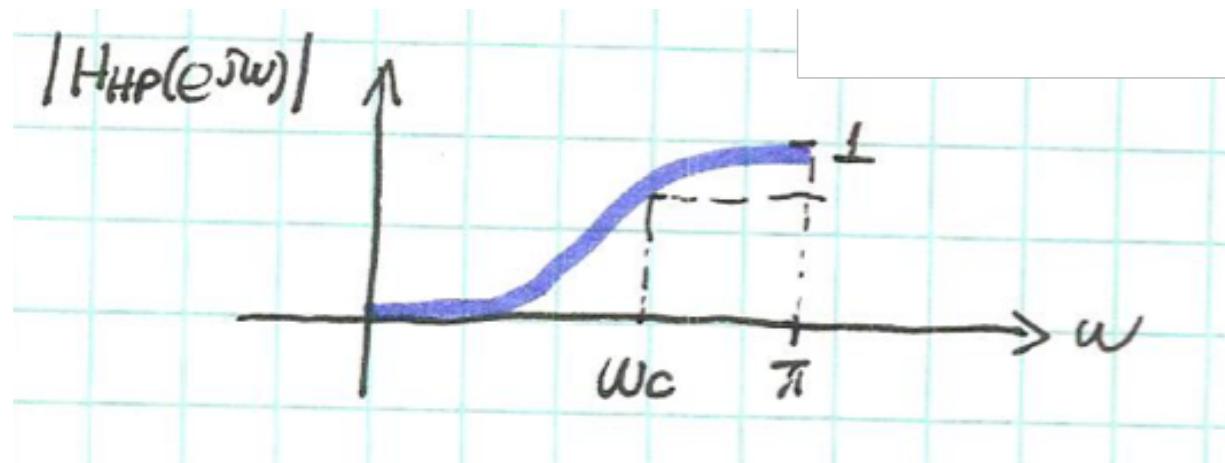


$\omega_c$  is the 3dB cutoff frequency

$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$

# Simple high pass filter

$$H_{HP}(z) = \frac{1 + \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$



$\omega_c$  is the 3dB cutoff frequency

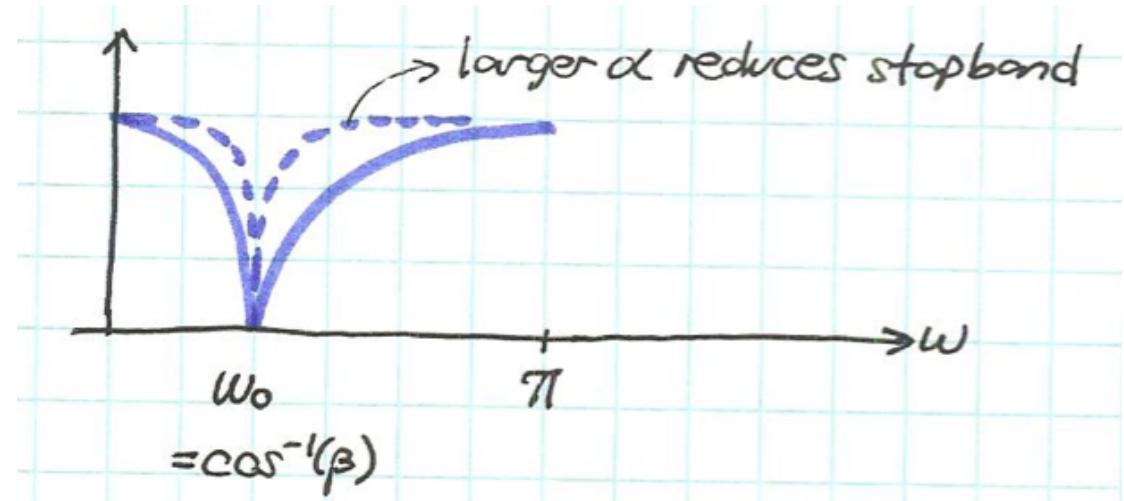
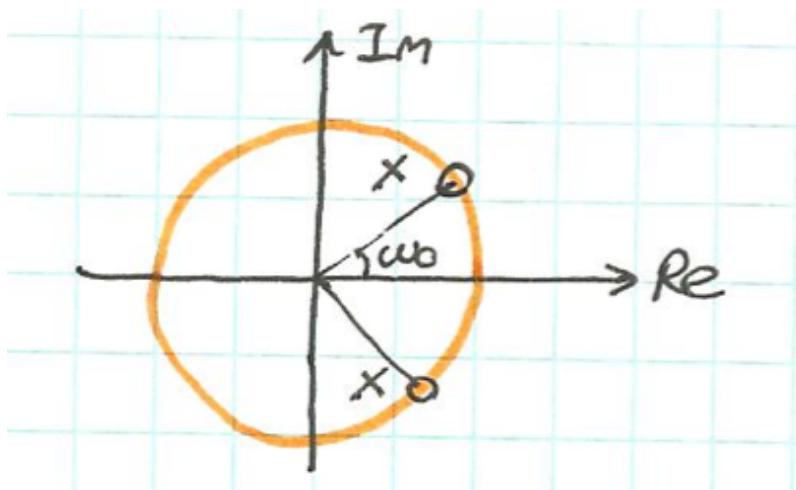
$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$

same as low pass

# Simple band-stop (Notch) filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad |\alpha| < 1 \quad |\beta| < 1$$

Note:  $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$   
 $\cos(\omega_0) = \beta$

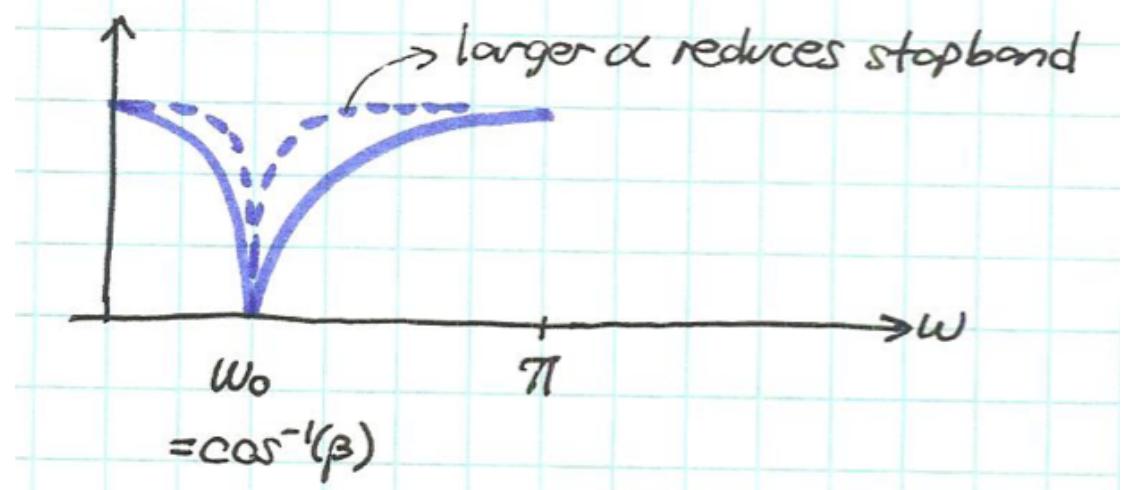
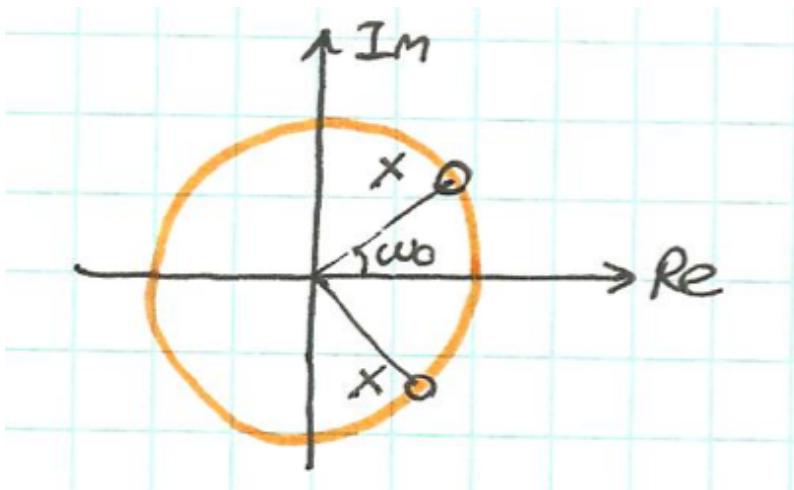


# Simple band-stop (Notch) filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad |\alpha| < 1 \quad |\beta| < 1$$

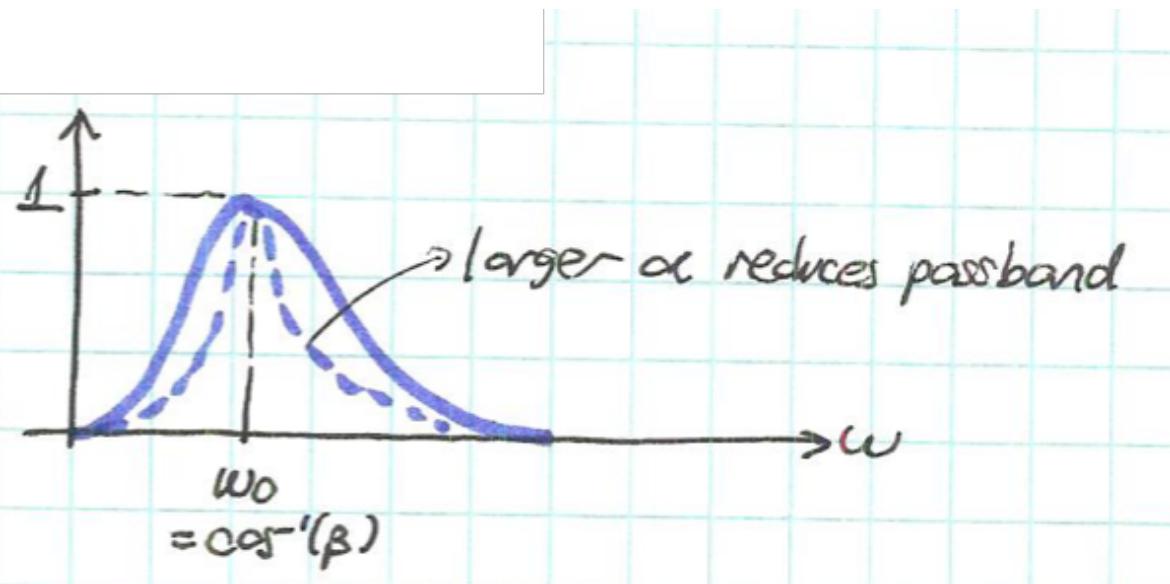
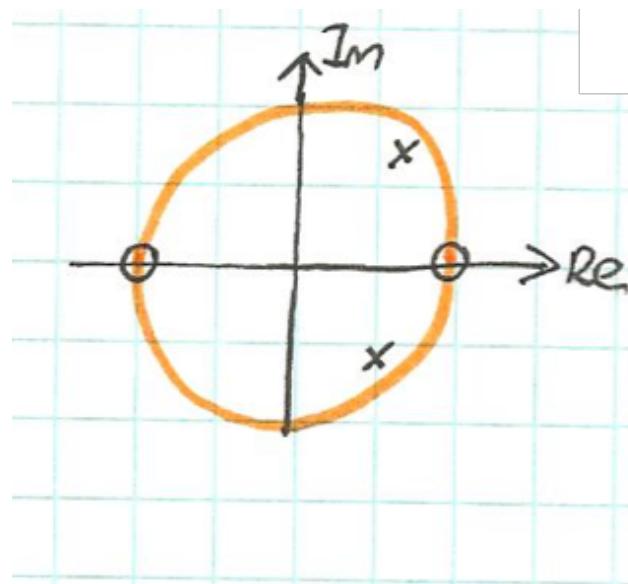
Note: As  $\alpha \rightarrow 1$  poles approach zeros

$$H_{BS}(\pm 1) = \frac{1 + \alpha}{2} \frac{2 \pm 2\beta}{(1 + \alpha)(1 \pm \beta)} = 1$$



# Simple band-pass filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad |\alpha| < 1 \quad |\beta| < 1$$



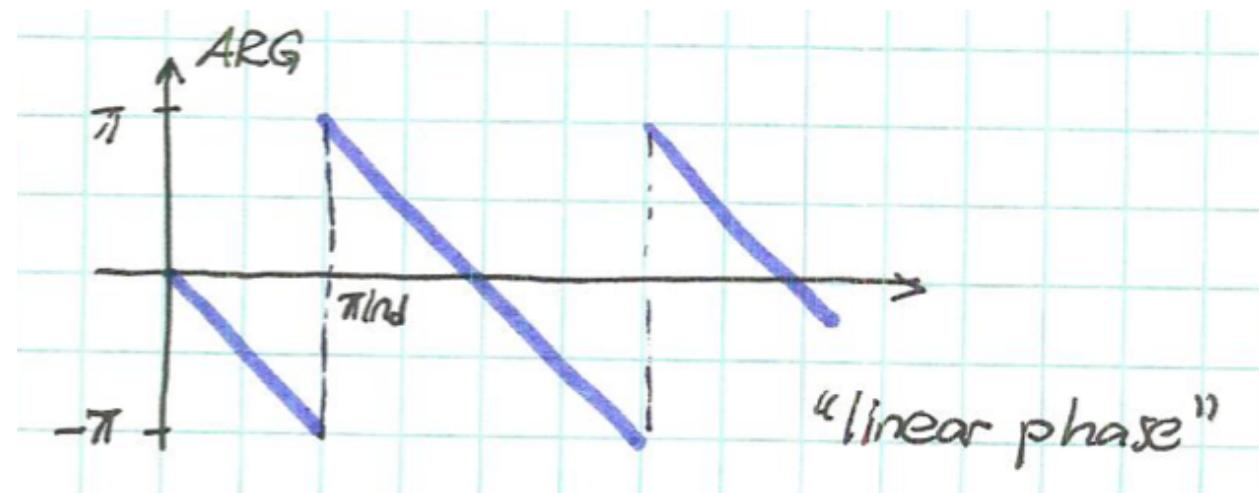
# Phase response

Example:  $H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

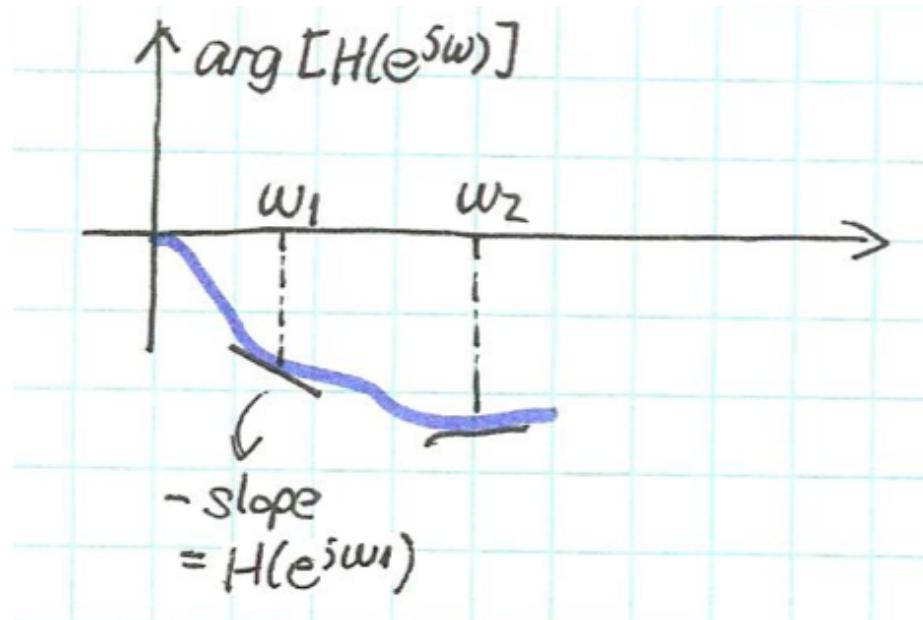
ARG is the wrapped phase  
arg is the unwrapped phase



# Group delay

To characterize general phase response, look at the group delay:

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

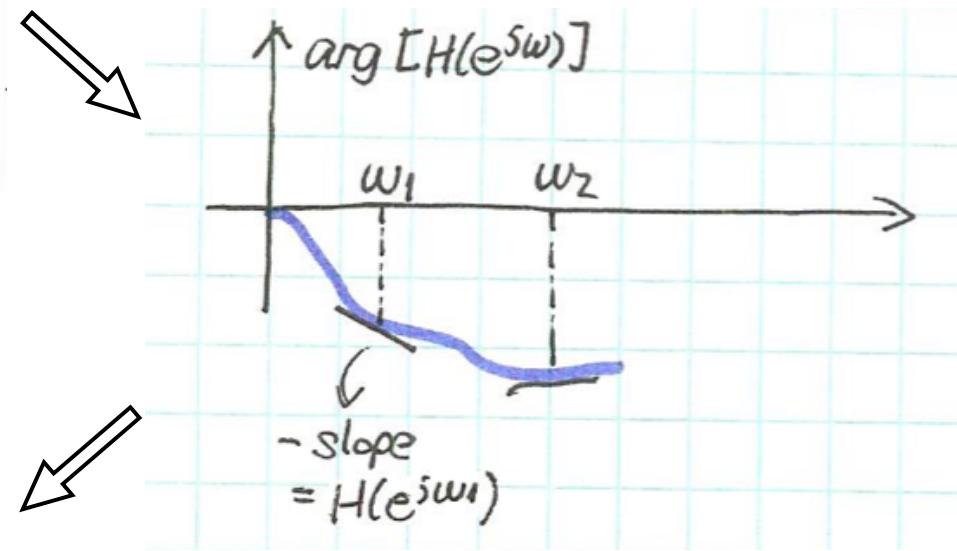
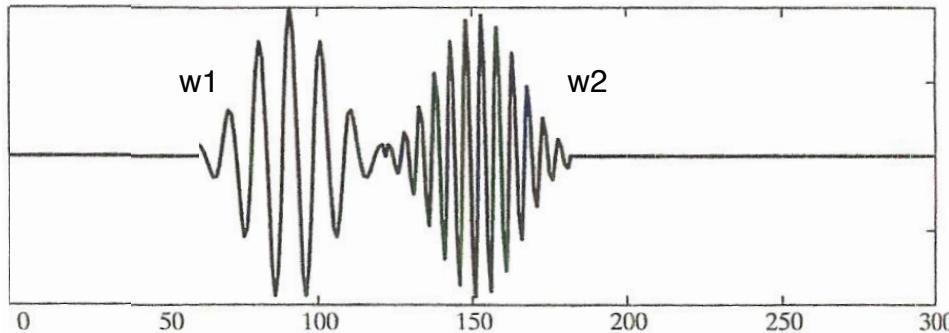


For linear phase system, the group delay is  $n_d$

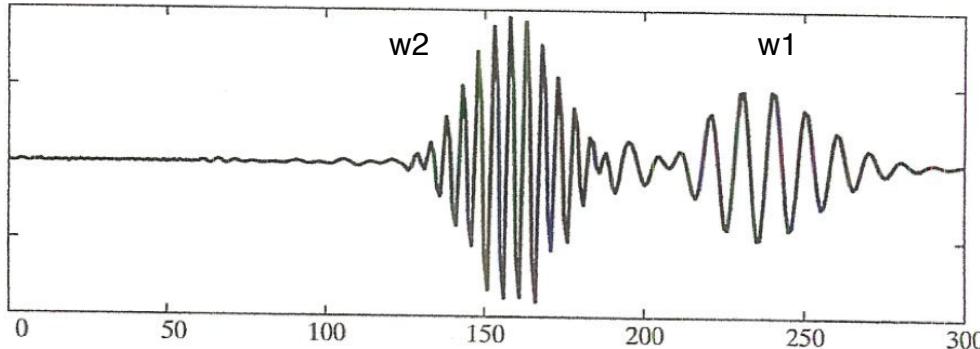
# Group delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Input



Output



For narrowband signals, phase response looks like a linear phase

## Group delay math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

arg of products is sum of args

$$\arg[H(e^{jw})] = -\sum_{k=1}^N \arg[1 - d_k e^{-jw}] + \sum_{k=1}^M \arg[1 - c_k e^{-jw}]$$

$$\text{grd}[H(e^{jw})] = -\sum_{k=1}^N \text{grd}[1 - d_k e^{-jw}] + \sum_{k=1}^M \text{grd}[1 - c_k e^{-jw}]$$

## Group delay math

$$\text{grd}[H(e^{j\omega})] = - \sum_{k=1}^N \text{grd}[1-d_k e^{-j\omega}] + \sum_{k=1}^M \text{grd}[1-c_k e^{-j\omega}]$$

Look at each factor:

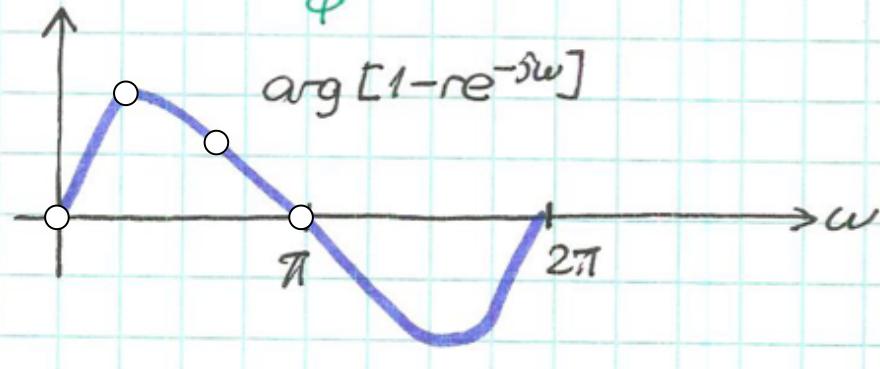
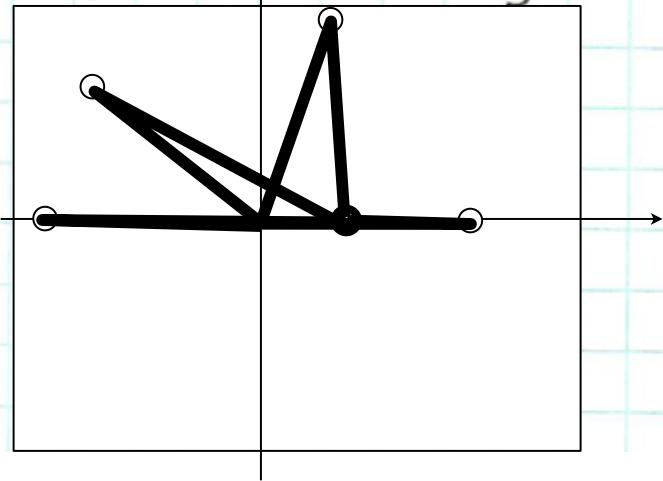
$$\arg[1 - \underbrace{r e^{j\theta}}_{c_k \text{ or } d_k} e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - r e^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - r e^{j\theta} e^{-j\omega}|^2}$$

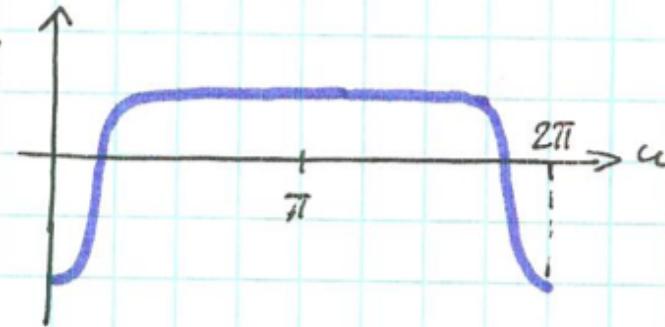
# Look at a zero lying on the real axis

Geometric Interpretation (for  $\theta=0$ )

$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega}-r]}_{\phi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

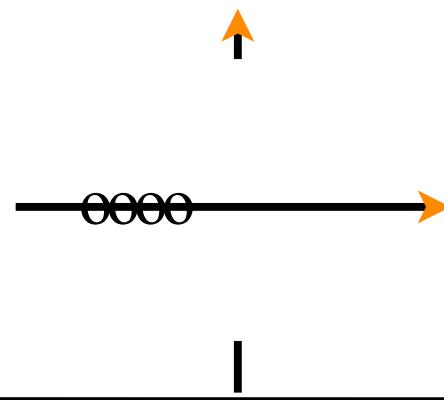
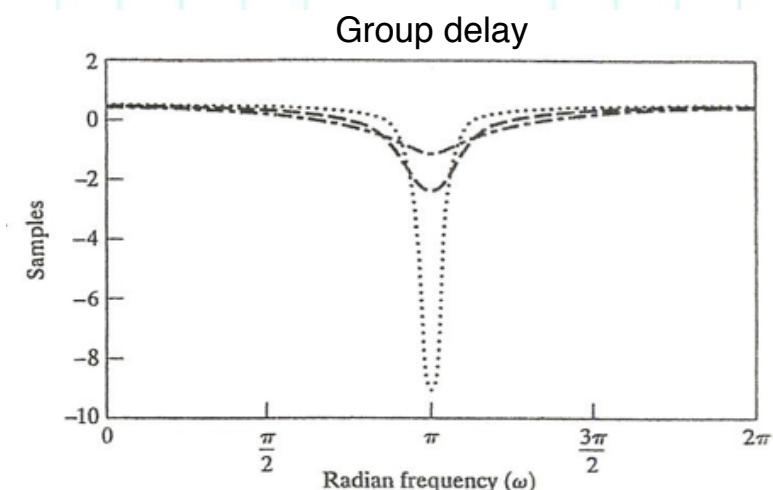
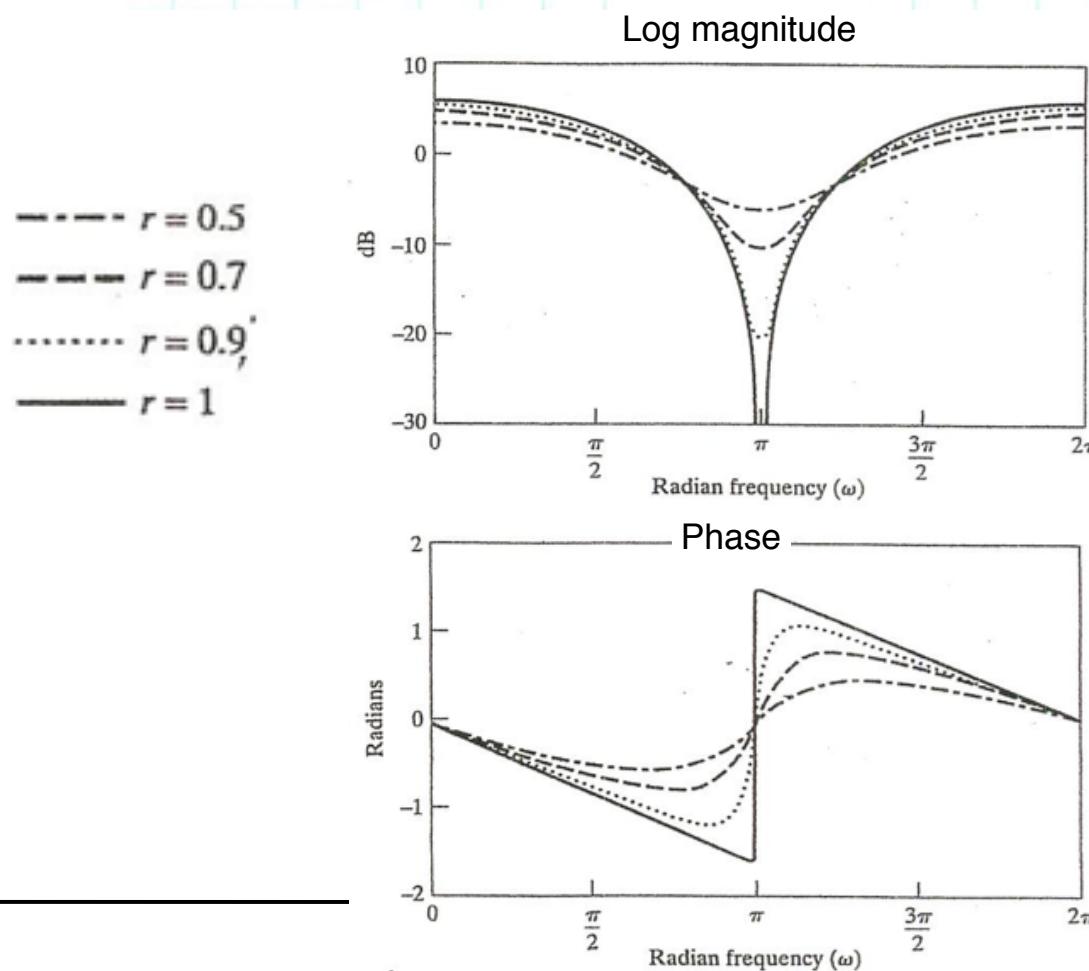


$$\text{grd}[1-re^{-j\omega}]$$



$\theta \neq 0 \Rightarrow$  shift to the right by  $\theta$

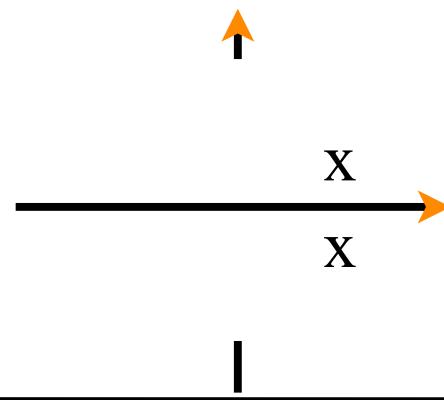
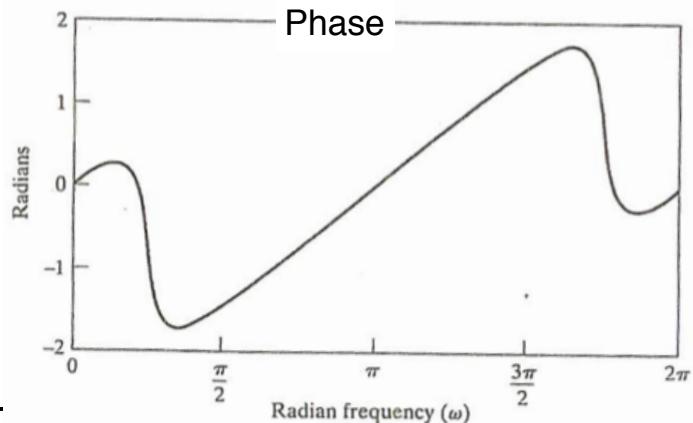
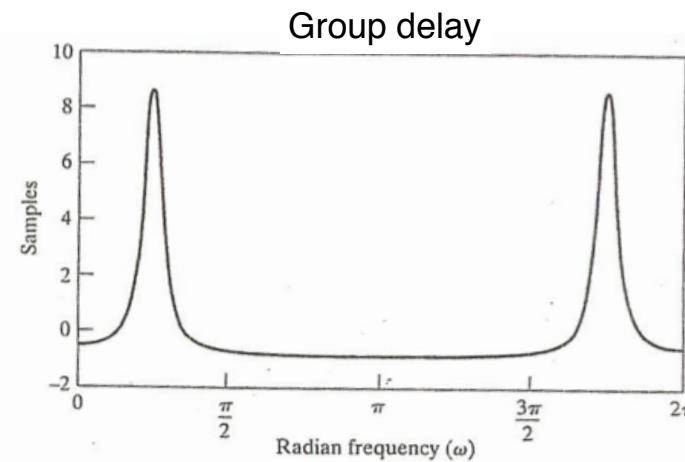
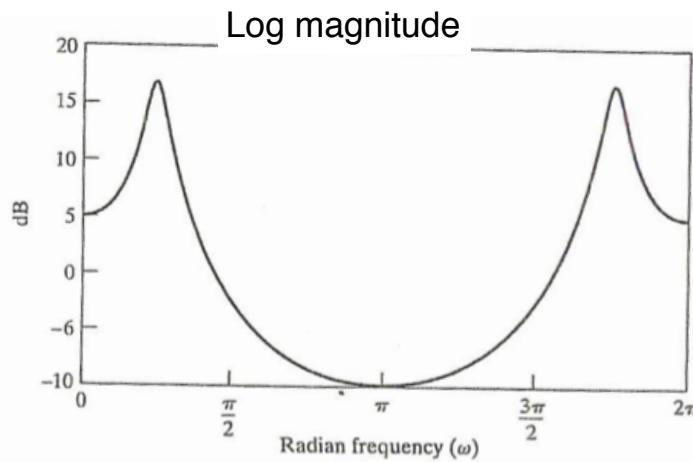
- \* Poles increase magnitude, but introduce phase lag and group delay.
- \* Zeros do the opposite.
- \* These effects are more marked when  $r \rightarrow 1$ .



## 2nd order IIR example

Example: 2nd order IIR with complex poles

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$



# 3rd order IIR example

Example: 3rd order IIR

