EE123 Discussion Section 3

Jon Tamir (based on slides by Frank Ong)

February 8, 2016

Announcements

- Submit Lab 1 by Thursday, Feb. 11 on bCourses
- Submit Homework 2 self-grade by Friday, Feb. 12 on bCourses
- Submit Homework 3 by Friday, Feb. 12 on bCourses
- Midterm 1 on Monday, Feb. 22 in class (2 hours)

Plan

- DFT problems
- DCT demo and problem
- FFT demo

The Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad \text{[Analysis]}$$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-kn} \qquad \text{[Synthesis]}$$

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Equivalent interpretations of the DFT:

- Sampling the DTFT at $\omega = \frac{2\pi}{N} k$
- The DTFT of periodically extended signal
- Sampling the z-transform at $z = e^{j\frac{2\pi}{N}k}$

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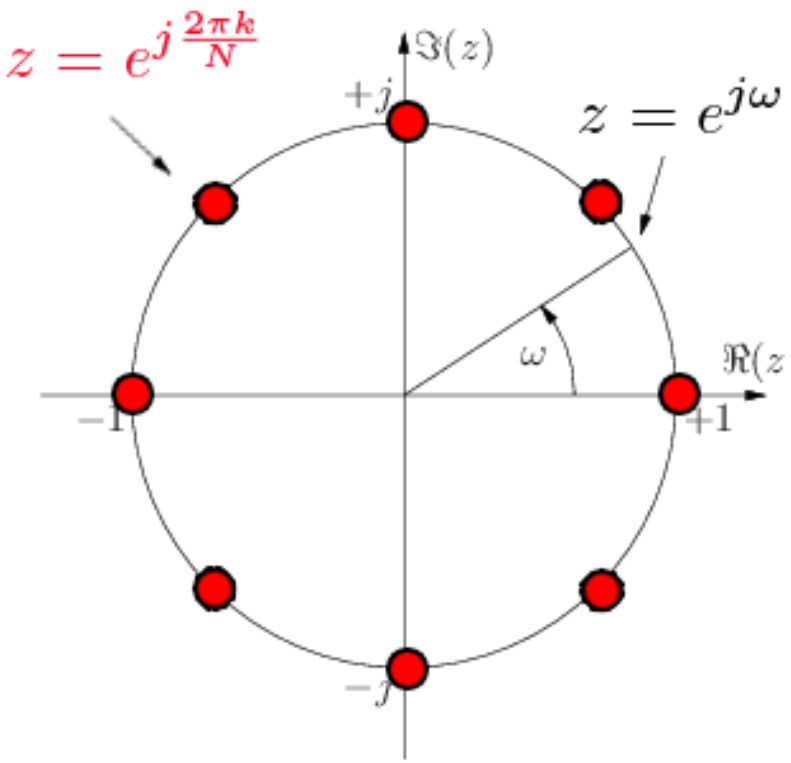
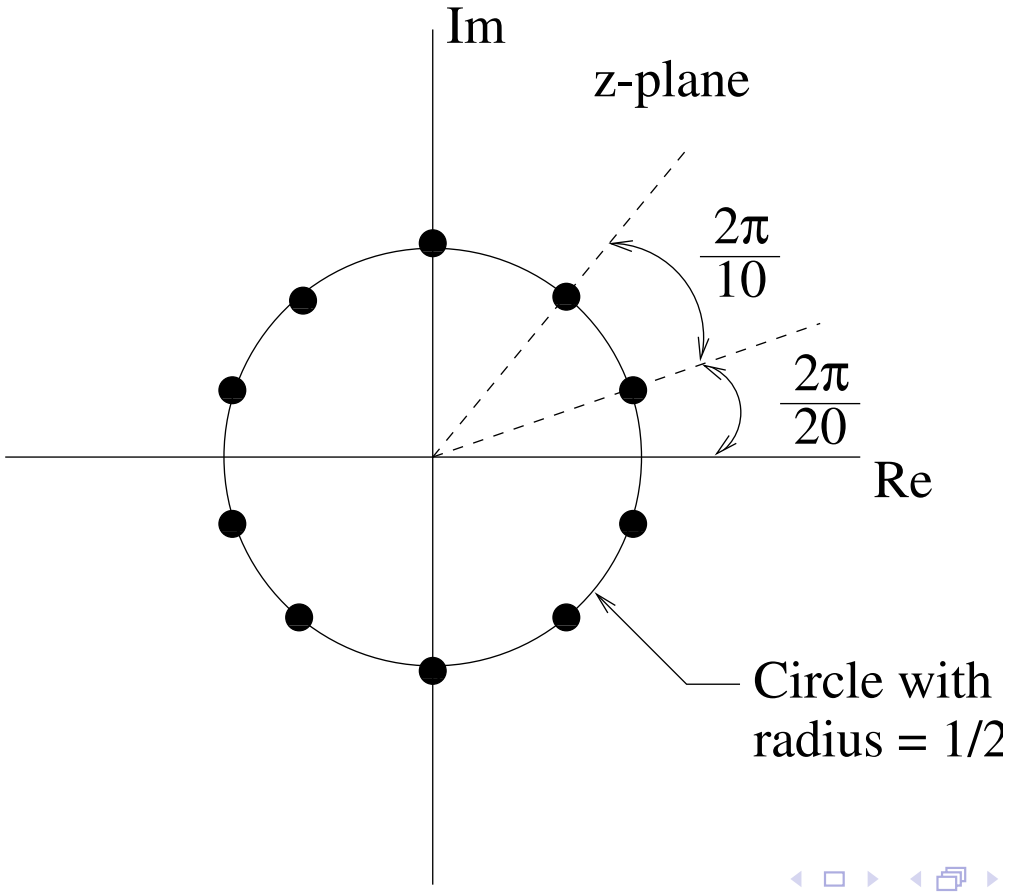
- Direct implementation: $\mathcal{O}(N^2)$
- Fast implementation: $\mathcal{O}(N \log N)$

The Discrete Fourier Transform

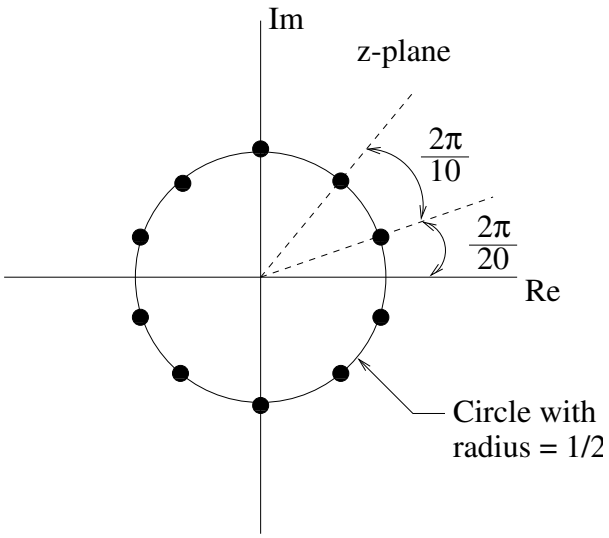


Question 1

Given a 10-point sequence $x[n]$, we wish to find equally spaced samples of its Z transform $X(z)$ on the contour as shown. Show how to achieve this using the DFT.

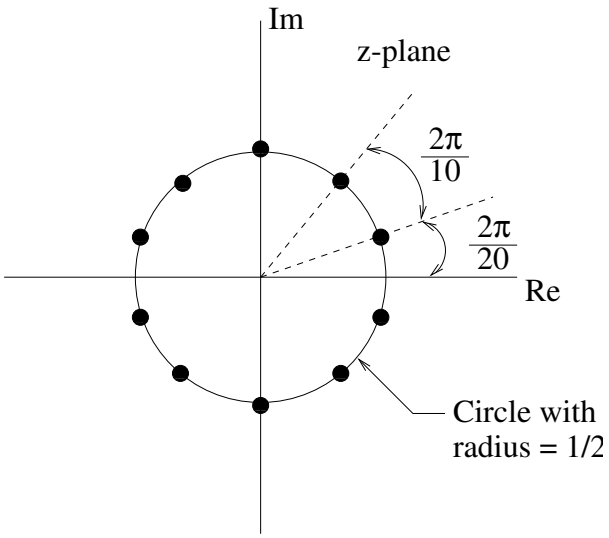


Solution 1



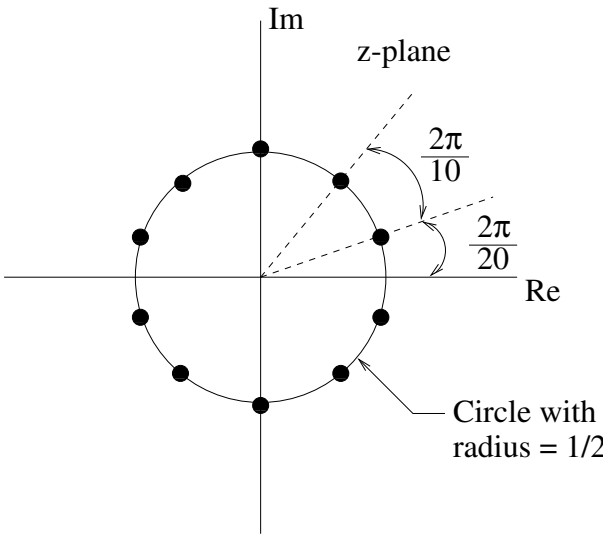
- Want to find $X(z)\big|_{z=\frac{1}{2}e^{j(2\pi k/10+\pi/10)}}$

Solution 1



- Want to find $X(z)|_{z=\frac{1}{2}e^{j(2\pi k/10+\pi/10)}}$
- $X(z)|_{z=\frac{1}{2}e^{j(2\pi k/10+\pi/10)}} = \sum_n x[n]0.5^{-n}e^{-j\frac{\pi n}{10}}e^{-j\frac{2\pi kn}{10}}$

Solution 1



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- $\implies DFT\{x[n]0.5^{-n}e^{-j\frac{\pi n}{10}}\}$

Question 2

$$x[n] = \{-3, 5, 4, -1, -9, -6, -8, 2\}$$

- a) Evaluate $\sum_{k=0}^{7}(-1)^k X[k]$
- b) Evaluate $\sum_{k=0}^{7} |X[k]|^2$

Solution 2

- part a)

$$(-1)^k = e^{-j\pi k} = e^{-j\frac{2\pi 4k}{8}} = W_8^{4k}$$

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$$\Rightarrow \sum_{k=0}^{7}(-1)^k X[k] = 8x[4] = -72$$

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$$x[4] = \frac{1}{8}\sum_{k=0}^{7} X[k]W_8^{4k}$$

$$\Rightarrow \sum_{k=0}^{7}(-1)^k X[k] = 8x[4] = -72$$

- part b) By Parseval's Theorem, $\sum_{k=0}^{7} |X[k]|^2 = 8\sum_{n=0}^{7} |x[n]|^2 = 1888.$

Question 3

$X[k]$ is the 9-point DFT of $x[n]$

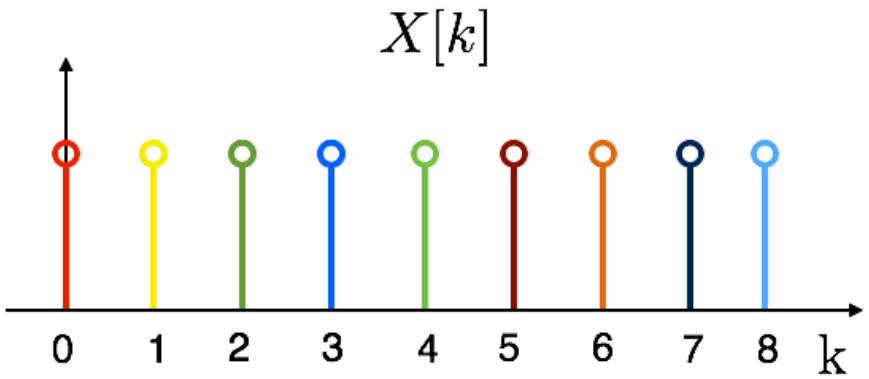
- Given:

  $$X[0] = -j, X[2] = 1-j, X[3] = 2-j, X[5] = 3-j, X[8] = 4-j$$

  If possible, determine the remaining 4 samples with the following assumptions:
- a) $x[n]$ is real
- b) $x[n]$ is conjugate symmetric

Solution 3

- a) $x[n]$ is real, then $X[k]$ is conjugate symmetric:



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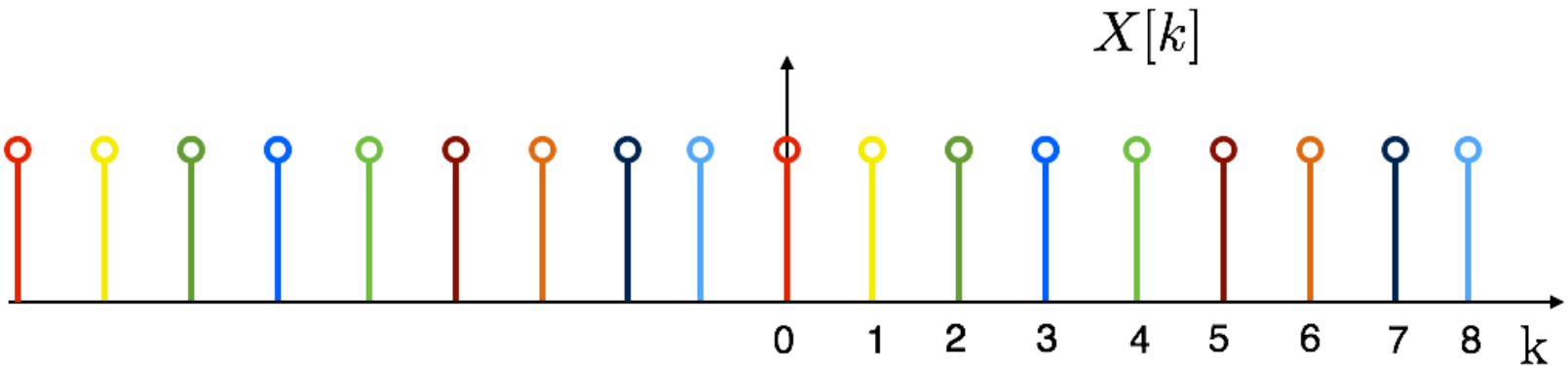

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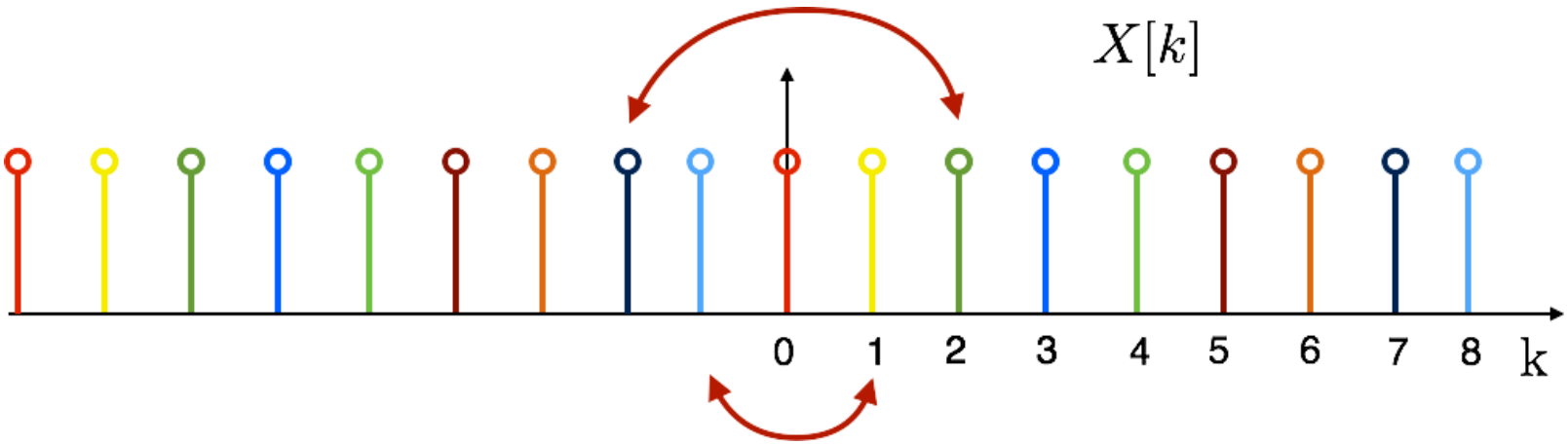

Solution 3

- a) $x[n]$ is real, then $X[k]$ is conjugate symmetric:

  $$X[1] = X^*[9-1] = 4+j, X[4] = 3+j, X[6] = 2+j, X[7] = 1+j$$

  BUT we also need $X[0] = X^*[0]$. Clearly $X[0]$ is not real, so this first condition cannot be met.
  Not Possible.

Solution 3

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  $$X[1] = X^*[9-1] = 4+j, X[4] = 3+j, X[6] = 2+j, X[7] = 1+j$$

  BUT we also need $X[0] = X^*[0]$. Clearly $X[0]$ is not real, so this first condition cannot be met.
  Not Possible.
- b) $x[n]$ is conjugate symmetric, then $X[k]$ is real. BUT this is not the case.
  Not possible

Question 4

Let $x[n]$ be some length-N sequence. Let $X[k] = DFT\{x[n]\}$

- Express $x_2[n] = DFT\{X[k]\}$ in terms of x[n]
- Hint: Try to match $DFT\{X[k]\}$ with the equation

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

Solution 4

$$x_2[n] = DFT\{X[k]\}$$

Solution 4

$$
\begin{aligned}
x_2[n] &= DFT\{X[k]\} \\
&= \sum_{k=0}^{N-1} X[k] W_N^{kn}
\end{aligned}
$$

Solution 4

$$
\begin{aligned}
x_2[n] &= DFT\{X[k]\} \\
&= \sum_{k=0}^{N-1} X[k] W_N^{kn} \\
&= \sum_{k=0}^{N-1} (X[-k])_N W_N^{-kn}
\end{aligned}
$$

Solution 4

$$
\begin{aligned}
x_2[n] &= DFT\{X[k]\} \\
&= \sum_{k=0}^{N-1} X[k] W_N^{kn} \\
&= \sum_{k=0}^{N-1} (X[-k])_N W_N^{-kn} \\
&= 8IDFT\{(X[-k])_N\}
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Solution 4

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&= \sum_{k=0}^{N-1} X[k] W_N^{kn} \\
&= \sum_{k=0}^{N-1} (X[-k])_N W_N^{-kn} \\
&= 8IDFT\{(X[-k])_N\} \\
&= 8(x[-n])_N
\end{aligned}
$$

Question 5

Let $x[n]$ be some length-N sequence. Let $X[k] = DFT\{x[n]\}$

- Express $x_3[k] = DFT\{DFT\{X[k]\}\}$ in terms of X[k]
- Express $x_4[n] = DFT\{DFT\{DFT\{X[k]\}\}\}$ in terms of x[n]

Solution 5

- Using the DFT properties:

$$
\begin{aligned}
x_2[n] &= 8(x[-n])_N \\
x_3[k] &= DFT\{x_2[n]\} = 8(X[-k])_N \\
x_4[n] &= DFT\{x_3[n]\} = 64x[n]
\end{aligned}
$$

Solution 5 continued

- Ignoring the scaling factors:

$$
\begin{aligned}
x_0[n] &= DFT^0\{x[n]\} = x[n] \\
x_1[k] &= DFT^1\{x[n]\} = X[k] \\
x_2[n] &= DFT^2\{x[n]\} = (x[-n])_N \\
x_3[k] &= DFT^3\{x[n]\} = (X[-k])_N \\
x_4[n] &= DFT^4\{x[n]\} = x[n]
\end{aligned}
$$

- The usual DFT operator is four-periodic. The nth power of the Fourier transform can be generalized as the **fractional Fourier transform**, which is used in optics.

Question 6: DCT

- The Discrete Cosine Transform (DCT) is a DFT-related transform that decomposes a finite signal in terms of a sum of cosine functions
- The DCT is often used in compression schemes, such as MP3, JPEG and MPEG.
- One of the reasons is its **energy compactness**

DCT Demo

- DCT Demo

Question 6: DCT

Wanting to know more why DCT performs much better than DFT, you decide to look closer at the definition of DCT

Given that the definition of the DCT (Type 2) is

$$X_c[k] = 2\sum_{n=0}^{N-1} x[n]\cos(\frac{\pi k(2n+1)}{2N})$$

Express $X_c[k]$ in terms of $X[k]$, the **2N**-point DFT of $x[n]$

Solution 6: DCT

$$X_c[k] = 2\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right)$$

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$$X_c[k] = 2\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right)$$
$$= \sum_{n=0}^{N-1} x[n]\left(e^{-j\frac{\pi k(2n+1)}{2N}} + e^{j\frac{\pi k(2n+1)}{2N}}\right)$$

Solution 6: DCT

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\begin{aligned}
X_c[k] &= 2\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k(2n+1)}{2N}\right) \\
&= \sum_{n=0}^{N-1} x[n]\left(e^{-j\frac{\pi k(2n+1)}{2N}} + e^{j\frac{\pi k(2n+1)}{2N}}\right) \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{2N}}e^{-j\frac{\pi k}{2N}} + \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi kn}{2N}}e^{j\frac{\pi k}{2N}}
\end{aligned}
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Solution 6: DCT

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&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{2N}}e^{-j\frac{\pi k}{2N}} + \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi kn}{2N}}e^{j\frac{\pi k}{2N}} \\
&= X[k]e^{-j\frac{\pi k}{2N}} + X[-k]e^{j\frac{\pi k}{2N}}
\end{aligned}
$$

Solution 6: DCT

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&= X[k]e^{-j\frac{\pi k}{2N}} + X[-k]e^{j\frac{\pi k}{2N}} \\
&= X[k]e^{-j\frac{\pi k}{2N}} + X^*[k]e^{j\frac{\pi k}{2N}}
\end{aligned}
$$

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&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{2N}}e^{-j\frac{\pi k}{2N}} + \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi kn}{2N}}e^{j\frac{\pi k}{2N}} \\
&= X[k]e^{-j\frac{\pi k}{2N}} + X[-k]e^{j\frac{\pi k}{2N}} \\
&= X[k]e^{-j\frac{\pi k}{2N}} + X^*[k]e^{j\frac{\pi k}{2N}} \\
&= 2\mathrm{Real}(X[k]e^{-j\frac{\pi k}{2N}})
\end{aligned}
$$

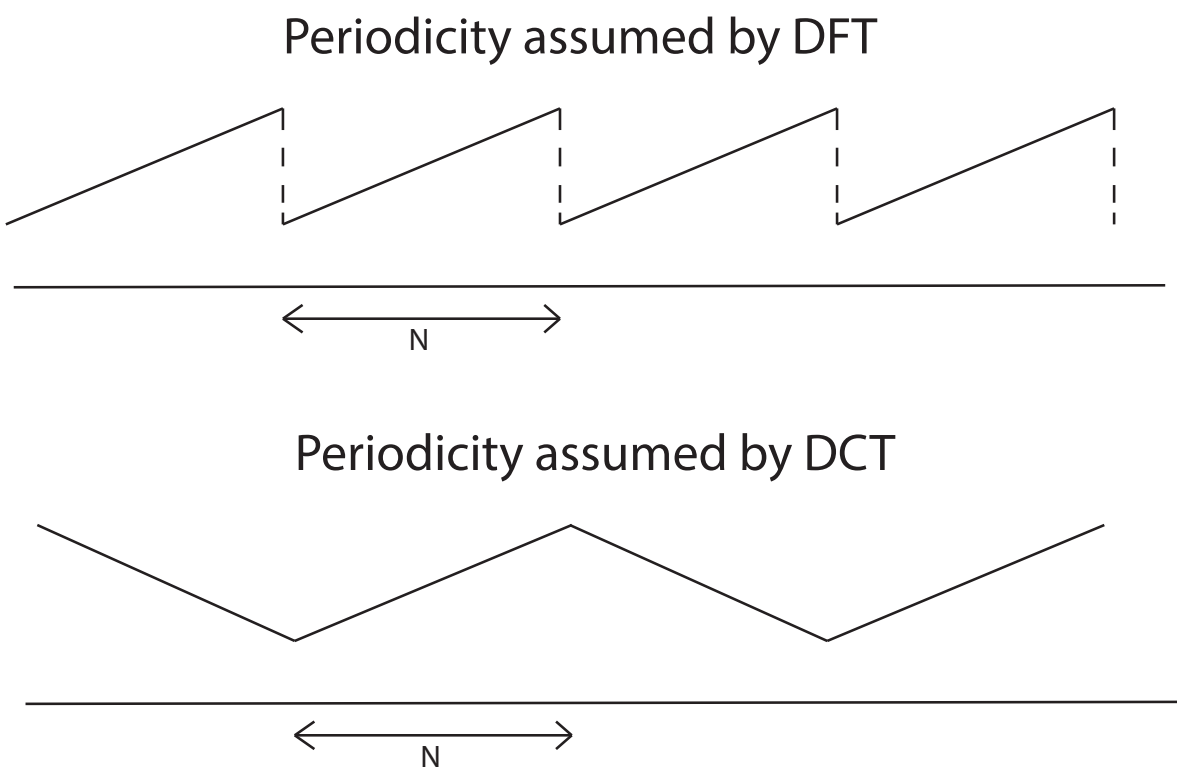
Solution 6: DCT

Key point is:

$$\begin{aligned} X_c[k] &= X[k]e^{-j\frac{\pi k}{2N}} + X[-k]e^{j\frac{\pi k}{2N}} \\ X_c[k] &= e^{-j\frac{\pi k}{2N}}(X[k] + X[-k]e^{j\frac{2\pi k}{2N}}) \end{aligned}$$

$$x_c[n] = \text{Shift}_{\frac{1}{2}}\{x[((n))_{2N}] + x[((-n-1))_{2N}]\}$$

Solution 6: DCT



DCT symmetric extension is better because sharp transitions require many coefficients to represent

FFT Demo

- FFT Demo