EE 123 Discussion Section 5

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Announcements

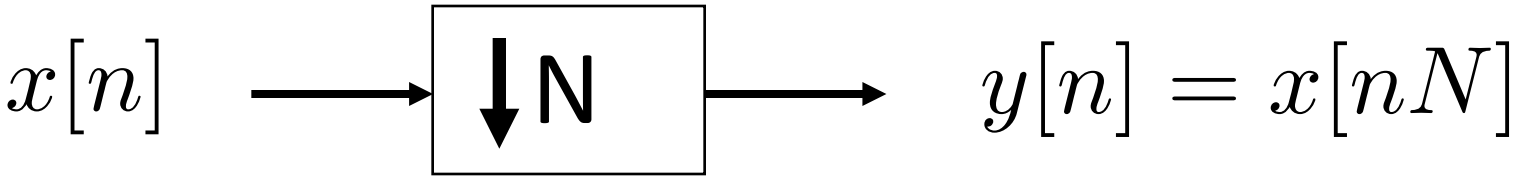
- HW 5 self grade – due Tuesday March 8
- Lab 3 – due Thursday March 10
- HW 6 – due Friday March 11

- Questions?

Plan

- Multi-rate DSP

Downsampling



- **Compresses** in the time domain
- **Expands** in the frequency domain

$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})$$

Alternative derivation of downsampling DTFT

- <u>Recall:</u> A N-periodic sequence has a discrete Fourier series (DFS):

$$\tilde{s}(m) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{S}_k e^{j2\pi mk/N}$$

DFS Representation

$$\tilde{S}_k = \sum_{m=0}^{N-1} \tilde{s}(m) e^{-j2\pi km/N}$$

DFS Coefficients

Alternative derivation of downsampling DTFT

- Impulse train: $$\tilde{s}(m) = \sum_{k \in \mathbb{Z}} \delta(m - kN)$$

- DFS Coefficients (check): $$\tilde{S}_k = 1$$

- DFS Representation: $$\tilde{s}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi mk/N}$$

Alternative derivation of downsampling DTFT

$$x[n] \longrightarrow \boxed{\downarrow N} \longrightarrow y[n] = x[nN]$$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} y[m]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]s[m]e^{-j\omega m/N},$$

$$s[m] = \begin{cases} 1 & \text{if } m = kN \\ 0 & \text{o.w.} \end{cases} = \sum_{k=-\infty}^{\infty} \delta(m - kN) = \frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi km/N}$$

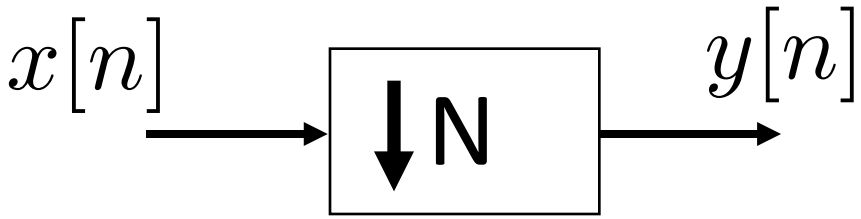
Alternative derivation of downsampling DTFT

$$x[n] \longrightarrow \boxed{\downarrow N} \longrightarrow y[n] = x[nN]$$

$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]e^{-j(\omega-2\pi k)m/N)}$$

$$\boxed{= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})}$$

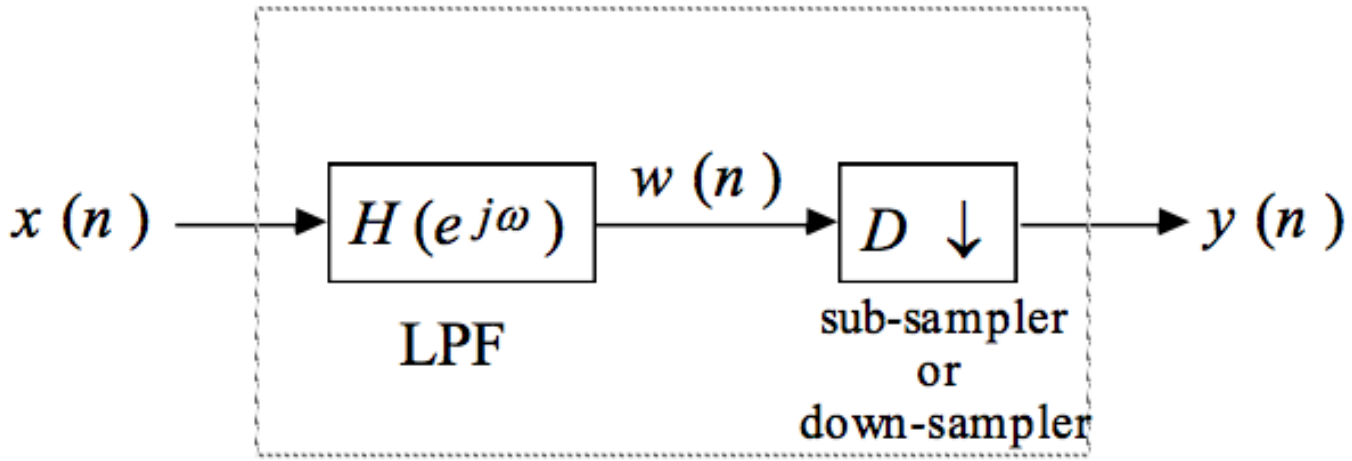
Downsampling



$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})$$

1. **Stretch** $X\left(e^{j\omega}\right)$ to $X(e^{\frac{j\omega}{N}})$
2. Create (N-1) copies of the stretched versions
3. Shift each copy by successive multiples of 2π and add
4. Divide by N

Decimation (filtering and downsampling)

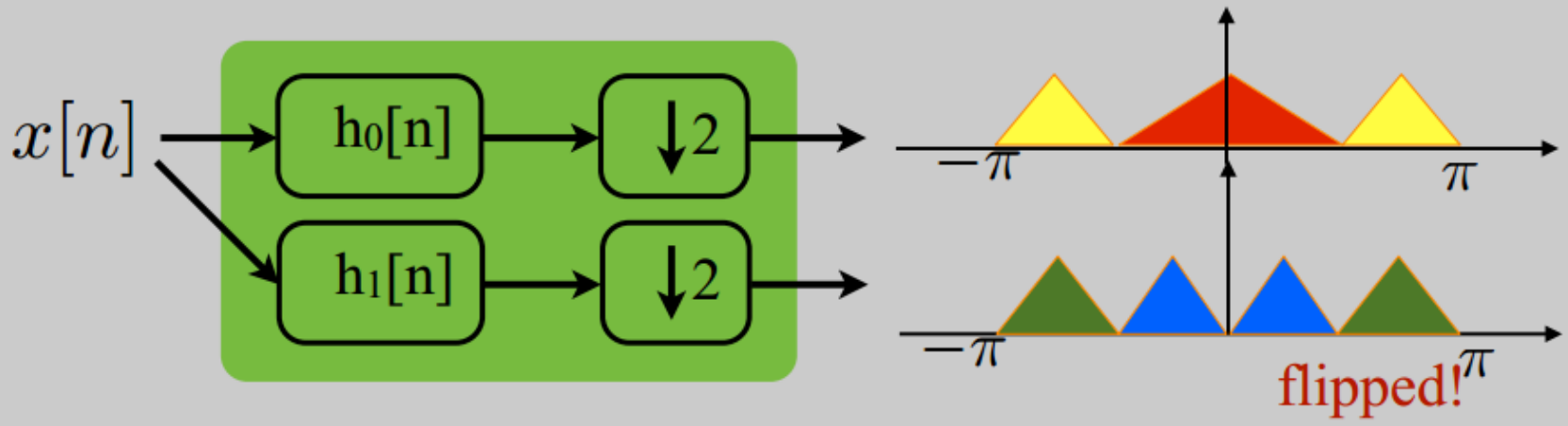


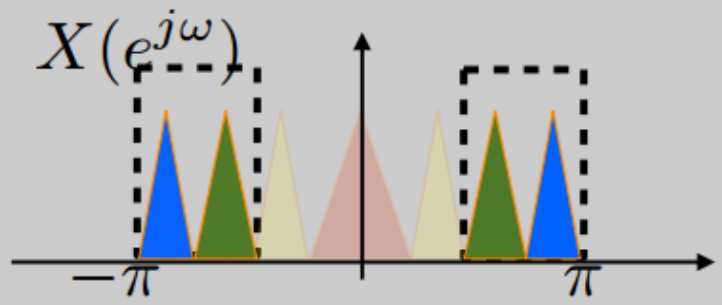
$$Y(e^{j\omega}) = \frac{1}{D} \sum_{p=0}^{D-1} H\left[e^{j\left(\frac{\omega - 2\pi p}{D}\right)}\right] X\left[e^{j\left(\frac{\omega - 2\pi p}{D}\right)}\right]$$

Decimation Demo

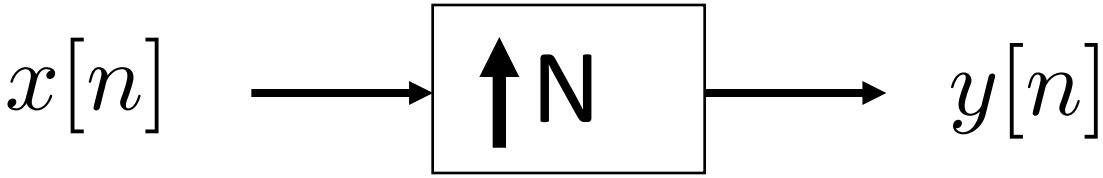
Subtleties in Time-Freq Tiling

- Assume h_0, h_1 are ideal low,high pass filters





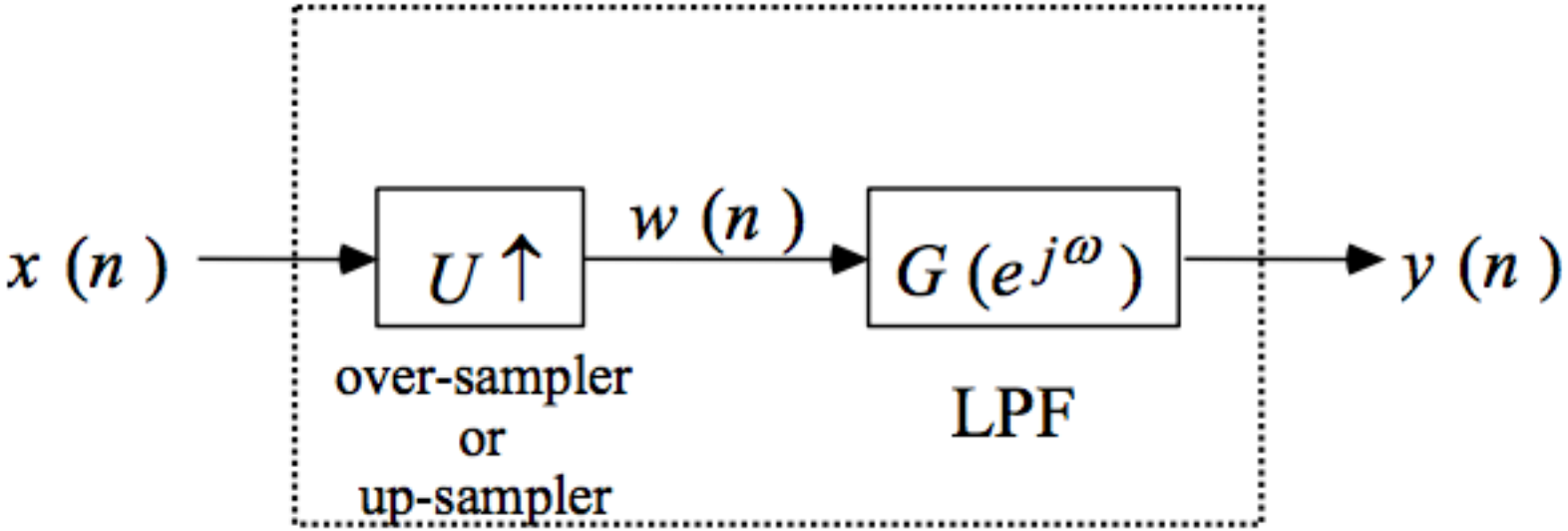
Upsampling



- **Expands** in the time domain
- **Compresses** in the frequency domain

$$Y(e^{j\omega}) = X(e^{j\omega N})$$

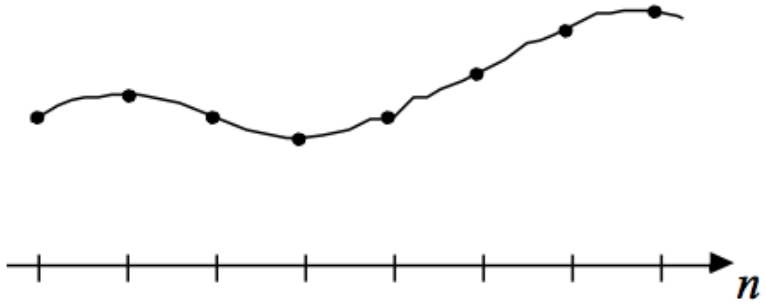
Interpolation (Upsampling and filtering)



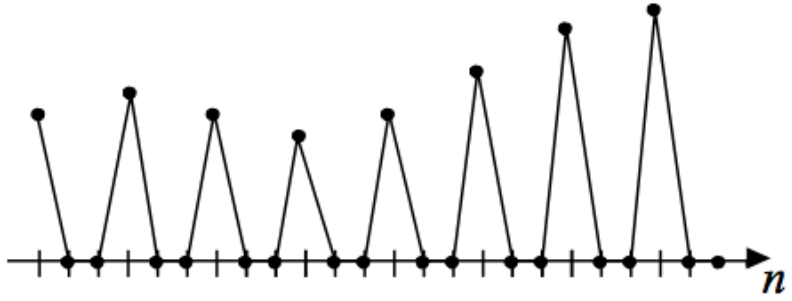
$$Y(e^{j\omega}) = G(e^{j\omega})X(e^{j\omega U})$$

Interpolation

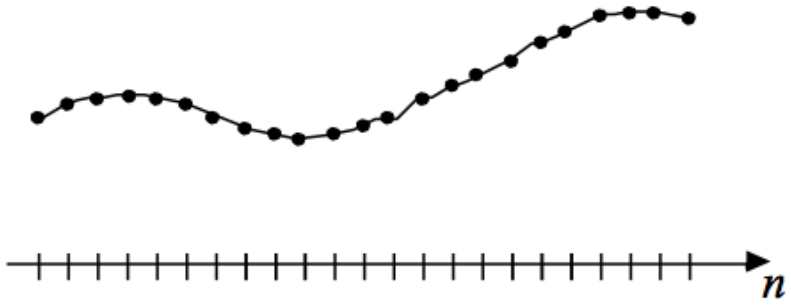
1. Smooth discrete-time signal
   - Low frequency content



2. Upsample by 3
   - No longer smooth! Contains high frequencies



3. LPF
   - Removes high frequencies



Interpolation Demo

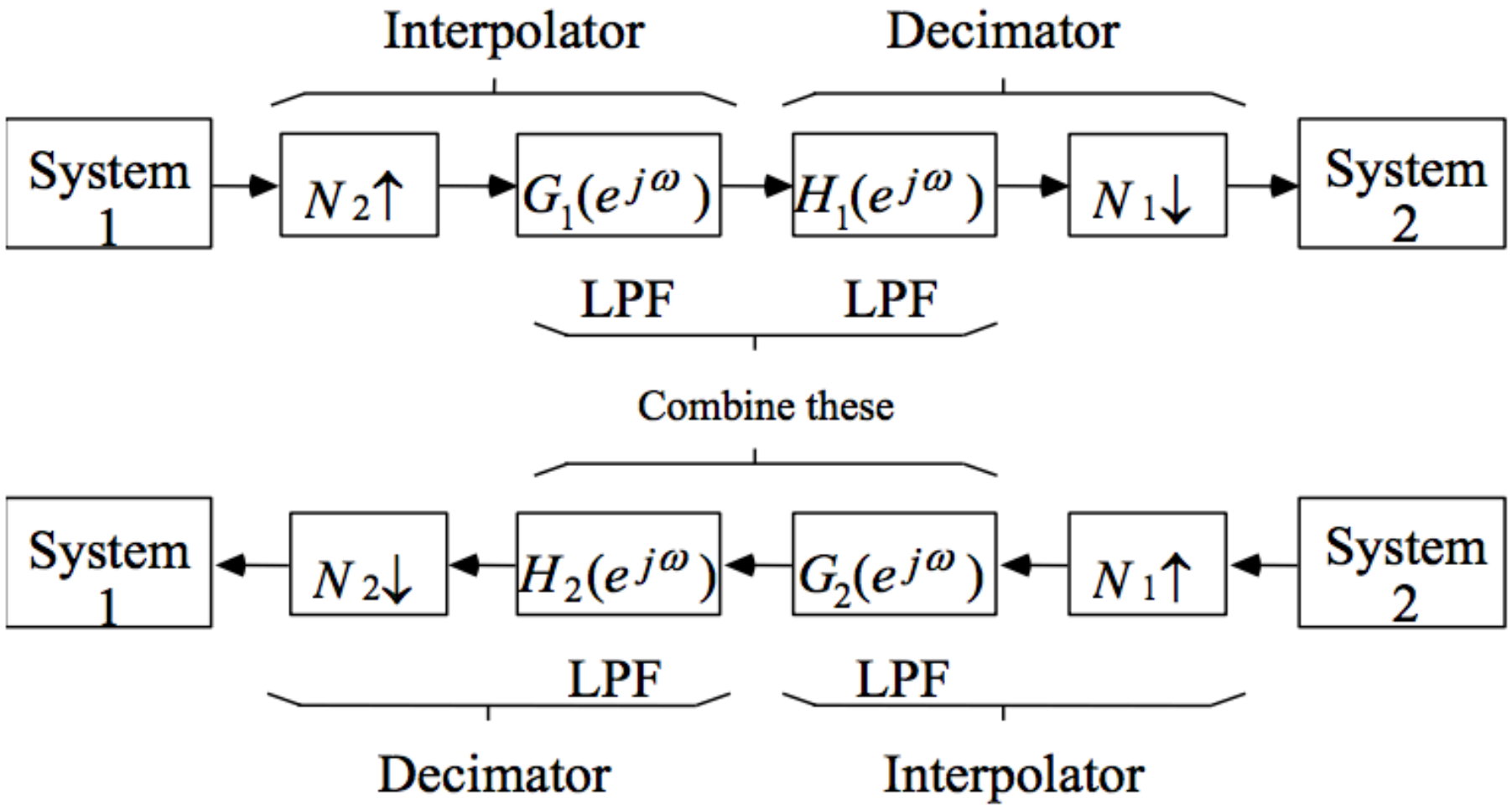
Sampling Rate Conversion

- Link two systems with different sampling rates

$$N = \frac{N_1}{N_2} = \frac{\text{System 1 Sampling Rate}}{\text{System 2 Sampling Rate}}$$

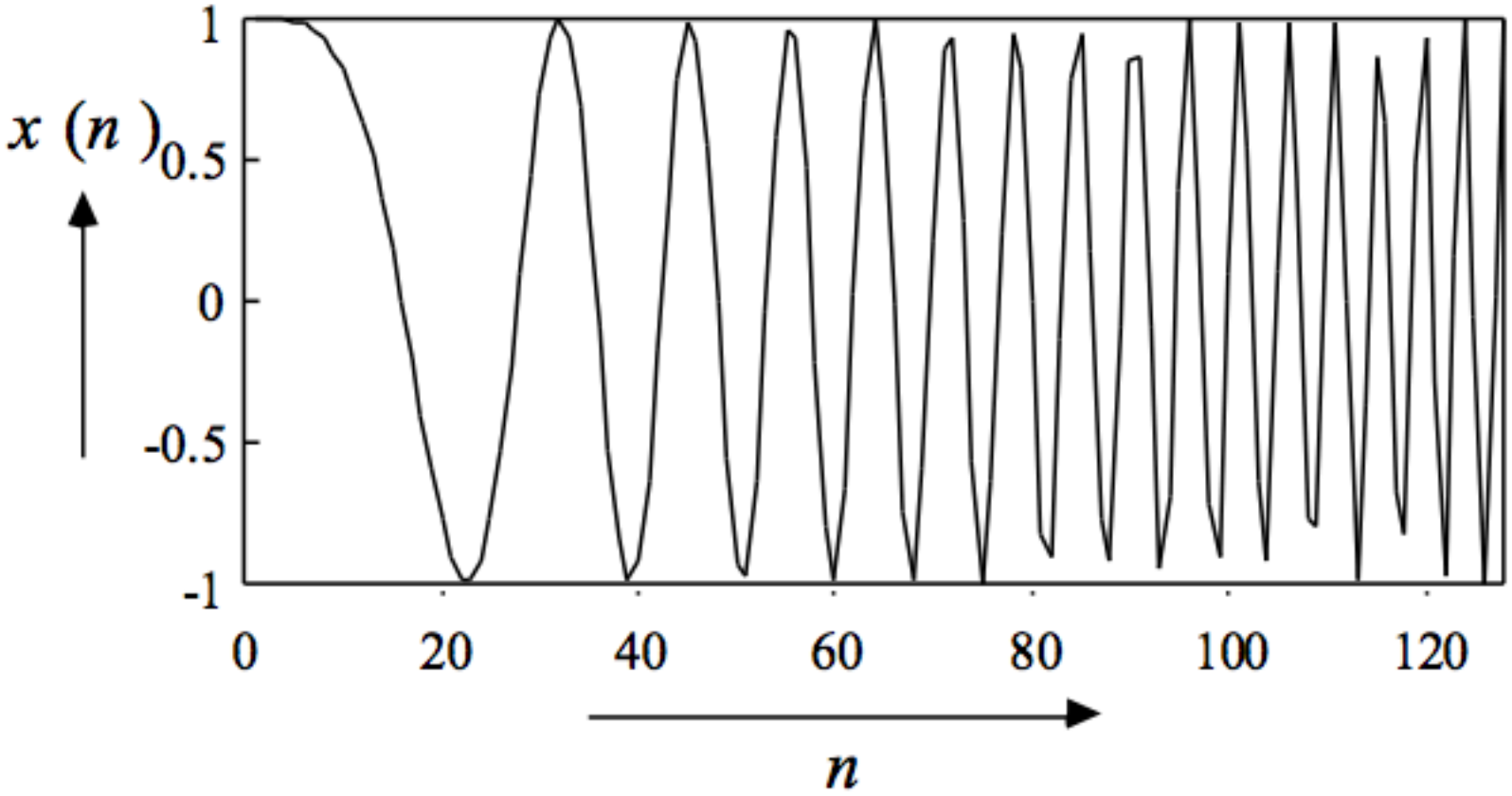
Sampling Rate Conversion

$$N = \frac{N_1}{N_2}$$



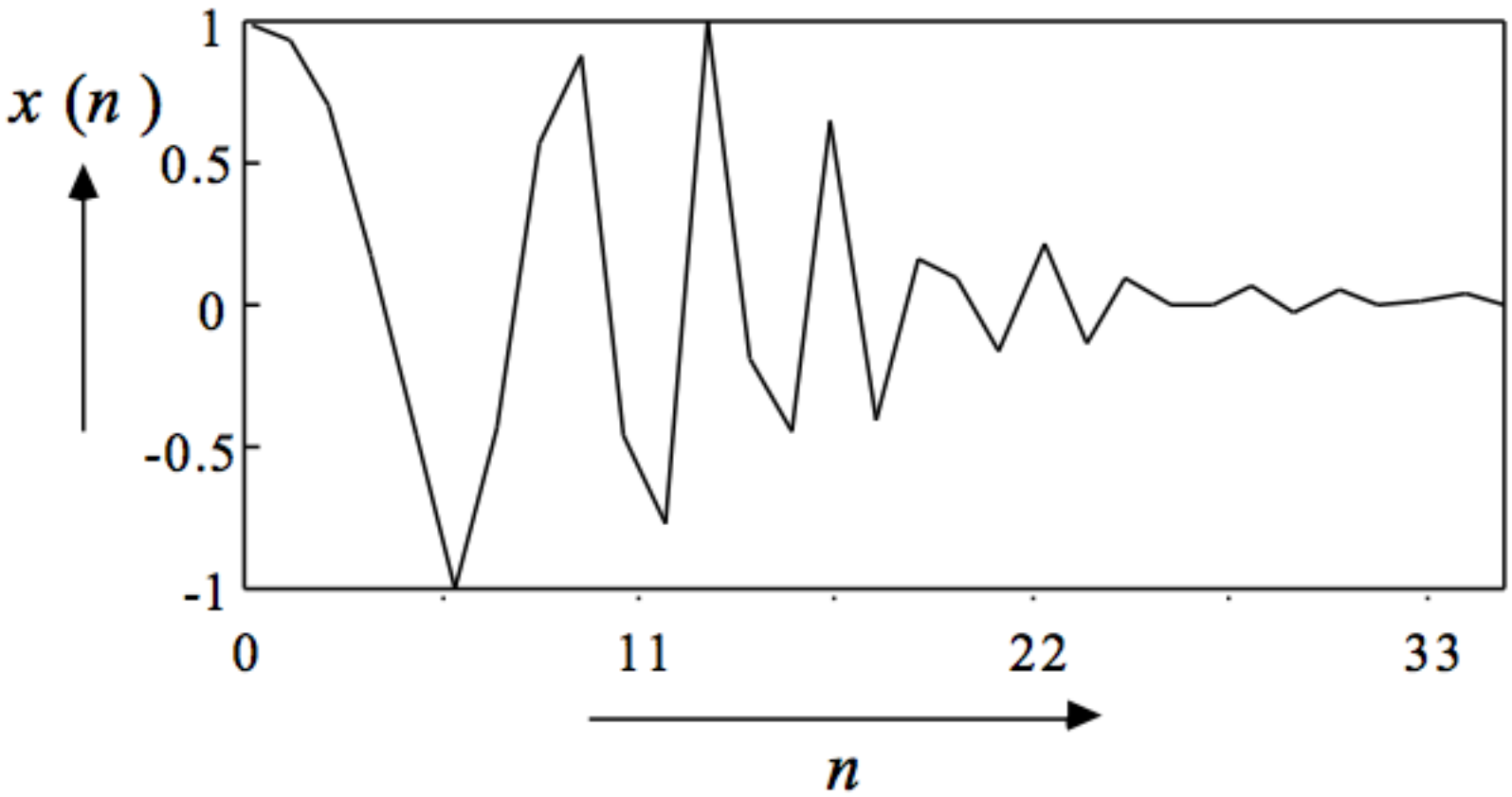
Sampling Rate Conversion

- Chirp



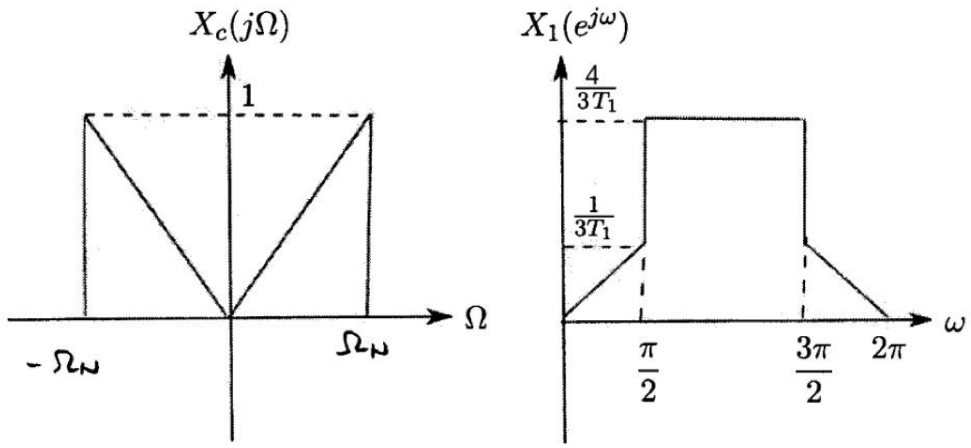
Sampling Rate Conversion

- Re-sample by a factor 11 / 3 (axis is rescaled)



Question 1

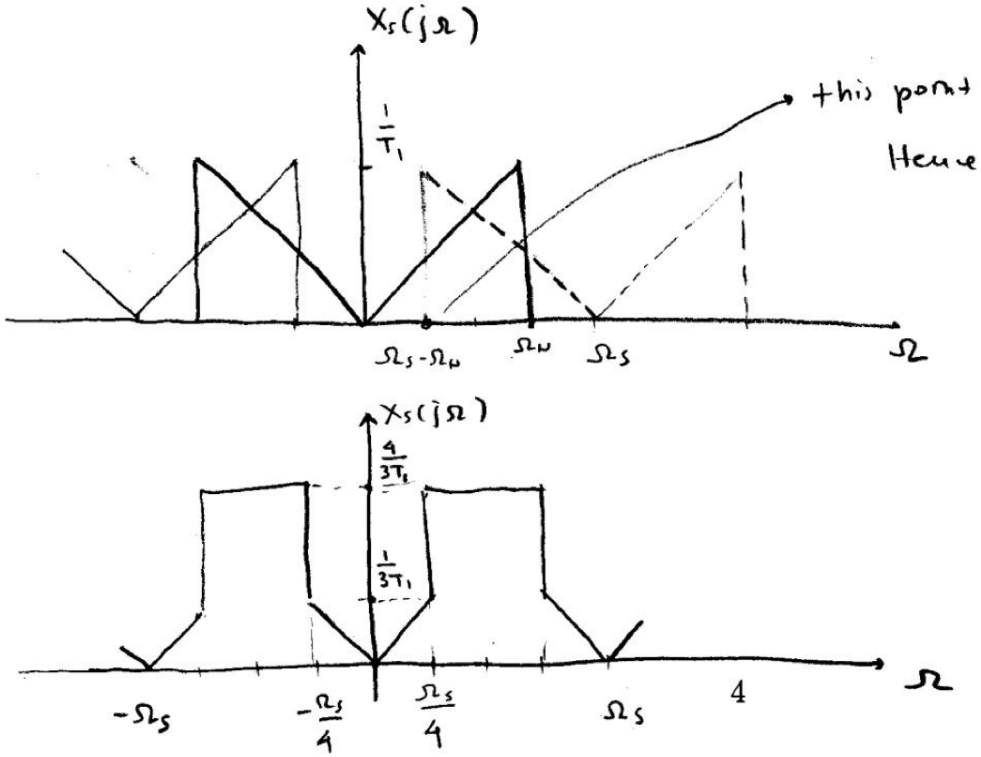
2. A continuous time signal $x_c(t)$ with the spectrum $X_c(j\Omega)$ depicted below is sampled with period T_1, resulting in a discrete sequence $x_1[n]$ with the DTFT $X_1(e^{j\omega})$ below.



a) (15 points) Determine the largest sampling period T_2 that would avoid aliasing, and express it in terms of T_1. Sketch the DTFT of the sequence $x_2[n]$, sampled with period T_2.

b) (15 points) Draw the block diagram of a post-processing unit that down-samples $x_2[n]$ by a factor of T_2/T_1. Sketch the DTFT of the output and compare it with $X_1(e^{j\omega})$ above.

Solution 1



this point maps to $\frac{\pi}{2}$.

Hence, $\Omega_s - \Omega_M = \frac{\Omega_s}{4}$

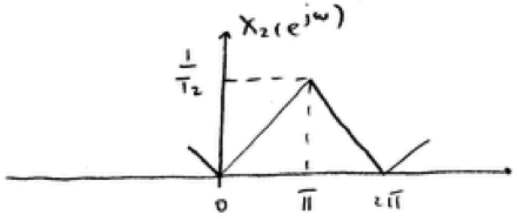
$\Rightarrow \Omega_s = \frac{4}{3}\Omega_M$

$\Omega_s = \frac{2\pi}{T_1}$

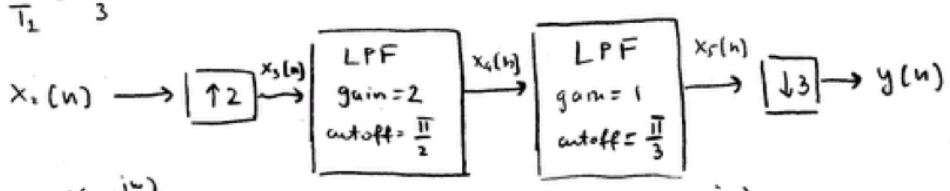
$\Rightarrow \boxed{\Omega_M = \frac{3}{4} \cdot \frac{2\pi}{T_1}}$

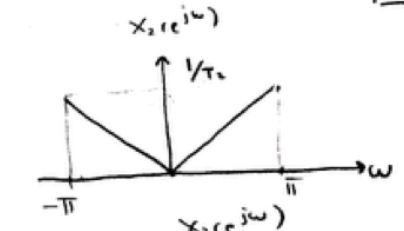
To avoid aliasing, we need

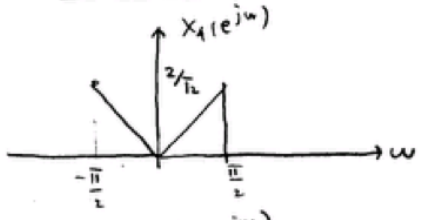
$$\frac{2\pi}{T_2} = 2\Omega_N = \frac{3\pi}{T_1} \Rightarrow \boxed{T_2 = \frac{2}{3}\cdot T_1}$$

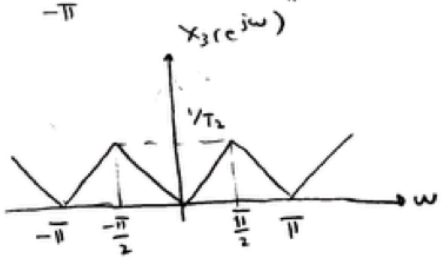


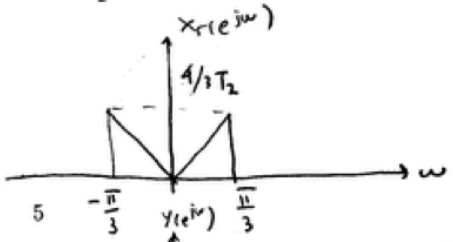
b) $\frac{T_2}{T_1} = \frac{2}{3}$

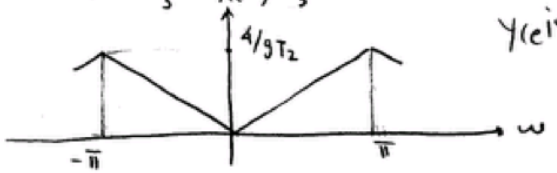












$Y(e^{j\omega}) \neq X_1(e^{j\omega})$

Question 2 a,b

4.46. Consider the system in Figure P4.46-1 with $H_0(z)$, $H_1(z)$, and $H_2(z)$ as the system functions of LTI systems. Assume that $x[n]$ is an arbitrary stable complex signal without any symmetry properties.

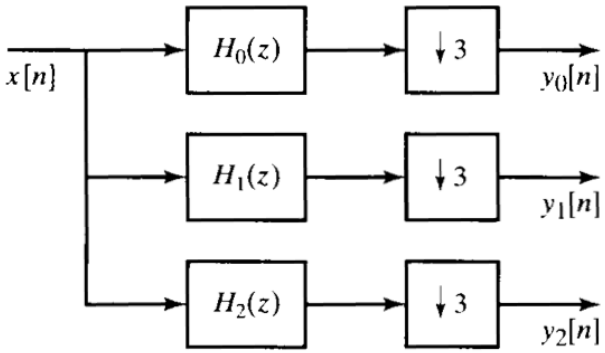


Figure P4.46-1

(a) Let $H_0(z) = 1$, $H_1(z) = z^{-1}$, and $H_2(z) = z^{-2}$. Can you reconstruct $x[n]$ from $y_0[n]$, $y_1[n]$, and $y_2[n]$? If so, how? If not, justify your answer.

(b) Assume that $H_0(e^{j\omega})$, $H_1(e^{j\omega})$, and $H_2(e^{j\omega})$ are as follows:

$$H_0(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/3, \\ 0, & \text{otherwise}, \end{cases}$$

$$H_1(e^{j\omega}) = \begin{cases} 1, & \pi/3 < |\omega| \le 2\pi/3, \\ 0, & \text{otherwise}, \end{cases}$$

$$H_2(e^{j\omega}) = \begin{cases} 1, & 2\pi/3 < |\omega| \le \pi, \\ 0, & \text{otherwise}. \end{cases}$$

Can you reconstruct $x[n]$ from $y_0[n]$, $y_1[n]$, and $y_2[n]$? If so, how? If not, justify your answer.

Solution 2 a,b

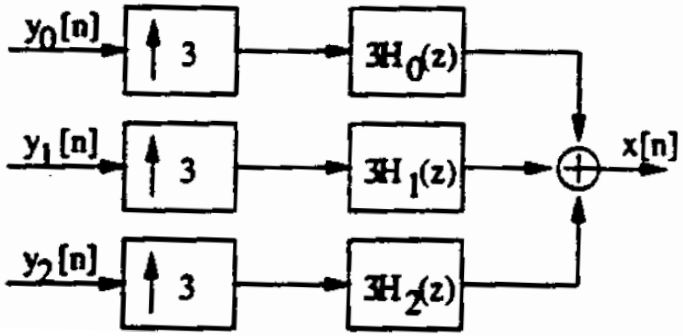
(a) Notice that

$$
\begin{aligned}
y_0[n] &= x[3n] \\
y_1[n] &= x[3n+1] \\
y_2[n] &= x[3n+2],
\end{aligned}
$$

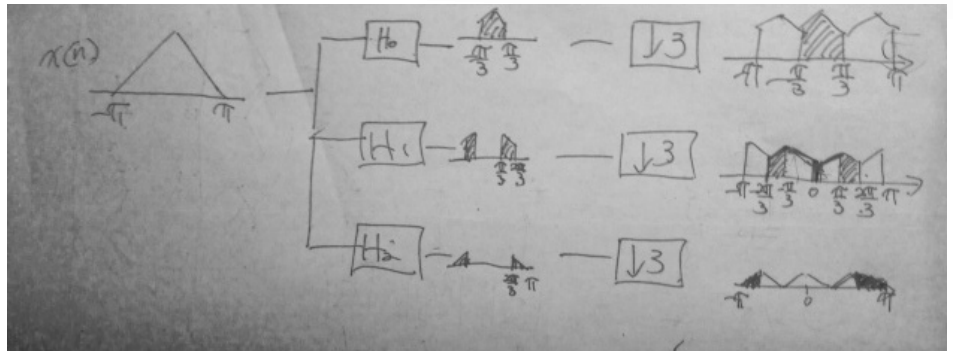
and therefore,

$$
x[n] = \begin{cases} y_0[n/3], & n = 3k \\ y_1[(n-1)/3], & n = 3k+1 \\ y_2[(n-2)/3], & n = 3k+2 \end{cases}
$$

(b) Yes. Since the bandwidth of the filters are $2\pi/3$, there is no aliasing introduced by downsampling. Hence to reconstruct $x[n]$, we need the system shown in the following figure:



This solution has an error. What is it?



Question 2c

Now consider the system in Figure P4.46-2. Let $H_3(e^{j\omega})$ and $H_4(e^{j\omega})$ be the frequency responses of the LTI systems in this figure. Again, assume that $x[n]$ is an arbitrary stable complex signal with no symmetry properties.

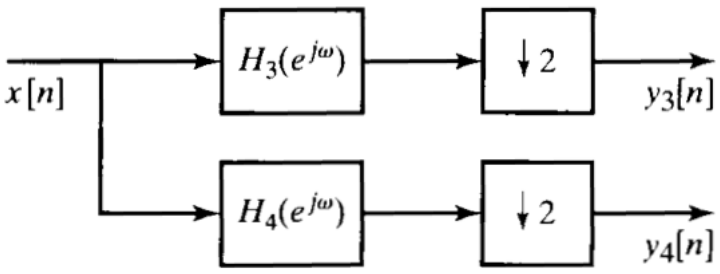


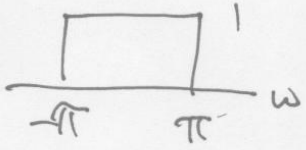
Figure P4.46-2

(c) Suppose that $H_3(e^{j\omega}) = 1$ and

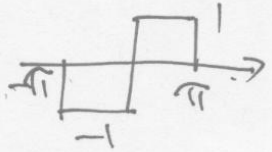
$$H_4(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega < \pi, \\ -1, & -\pi \le \omega < 0. \end{cases}$$

Can you reconstruct $x[n]$ from $y_3[n]$ and $y_4[n]$? If so, how? If not, justify your answer.

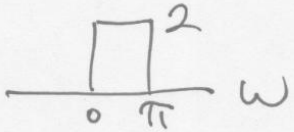
$H_3(e^{j\omega})$



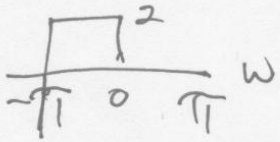
$H_4(e^{j\omega})$

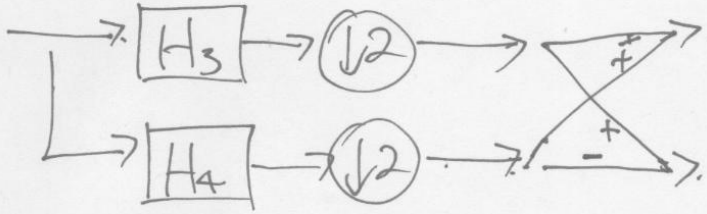


$H_3 + H_4$:

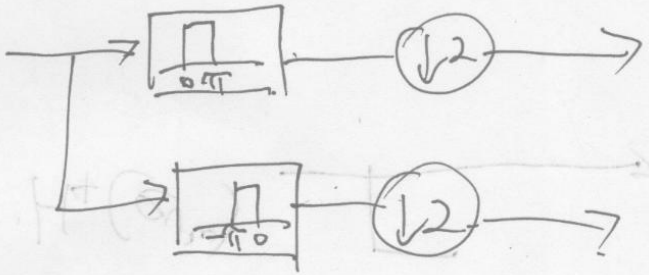


$H_3 - H_4$:



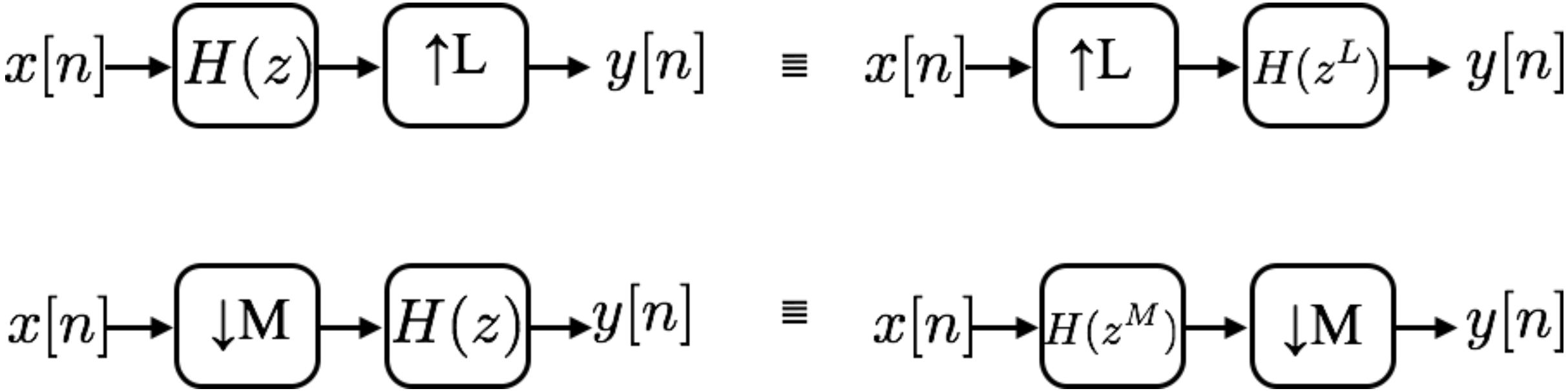


Sampling is LINEAR!



which we know how to reconstruct.

Multirate Noble Identities



Multirate Noble Identities

$$x[n] \rightarrow \boxed{\downarrow \mathbf{M}} \rightarrow \boxed{H(z)} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow \mathbf{M}} \rightarrow \tilde{y}[n]$$

- Proof for M=2:

Subsampling & filtering

"Pf": Ex. $x_n \rightarrow (2\downarrow) \rightarrow w_n \rightarrow \boxed{H(z)} \rightarrow y_n$

$$W(z) = \frac{1}{2}X(z^{1/2}) + \frac{1}{2}X(-z^{1/2})$$

$$Y(z) = \frac{1}{2}X(z^{1/2})H(z) + \frac{1}{2}X(-z^{1/2})H(z)$$

$x_n \rightarrow \boxed{H(z^2)} \rightarrow t_n \rightarrow (2\downarrow) \rightarrow y'_n$

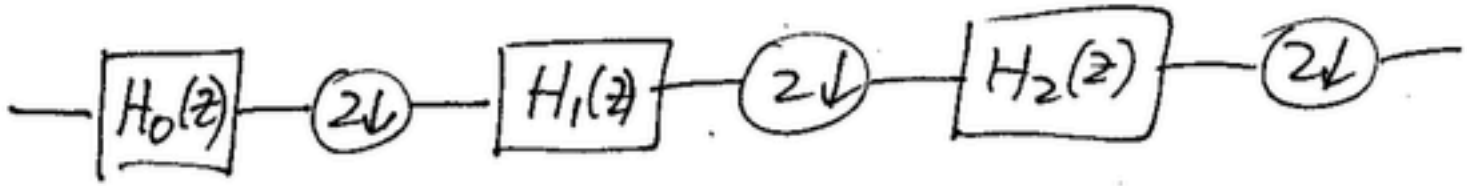
$$T(z) = X(z)H(z^2)$$

$$Y'(z) = \frac{1}{2}\left[T(z^{1/2}) + T(-z^{1/2})\right]$$

$$= \frac{1}{2}X(z^{1/2})H(z) + \frac{1}{2}X(-z^{1/2})H(z)$$

$$= Y(z)$$

Ex. Cascade of subsampling and filtering



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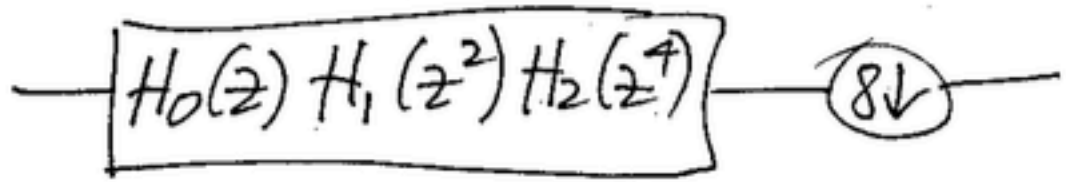

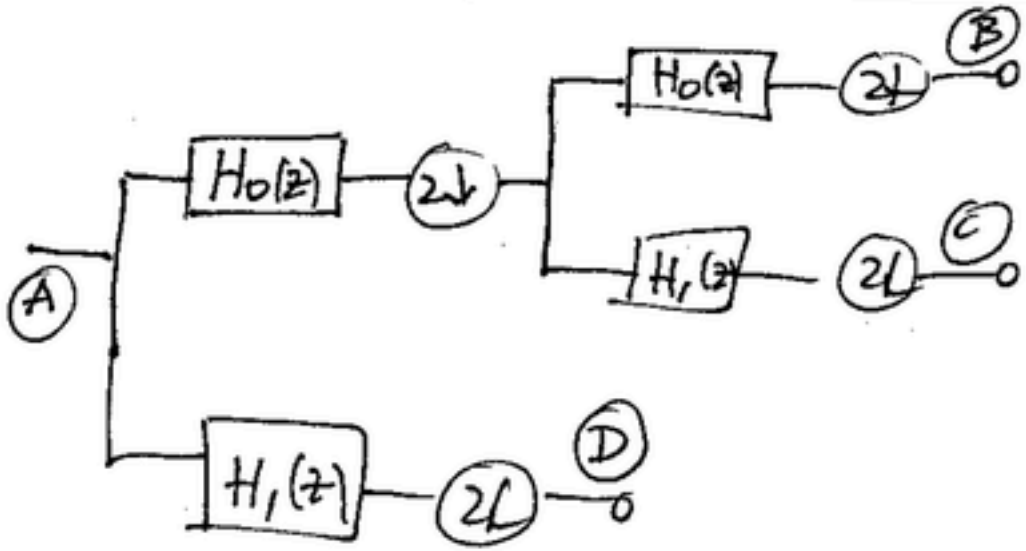

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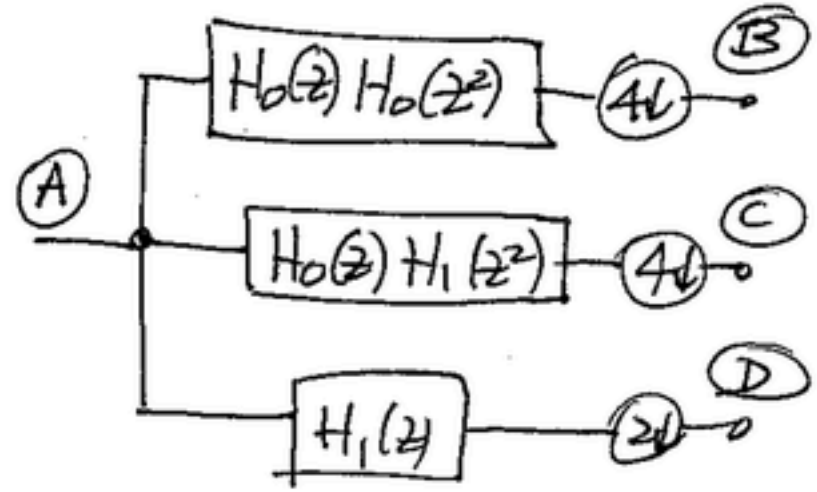
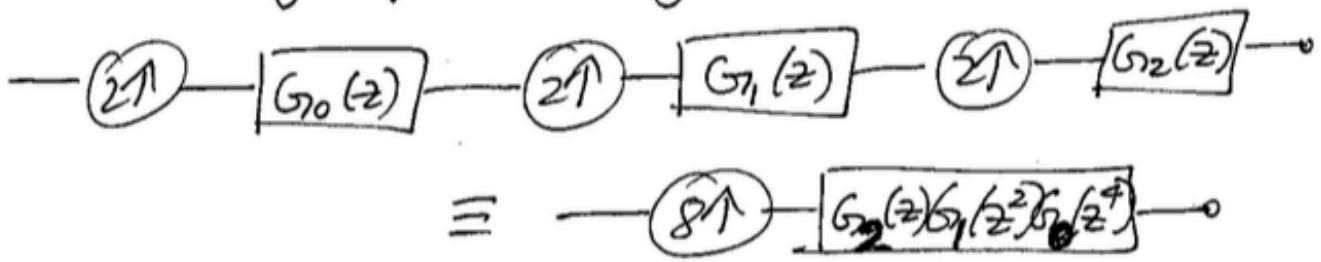
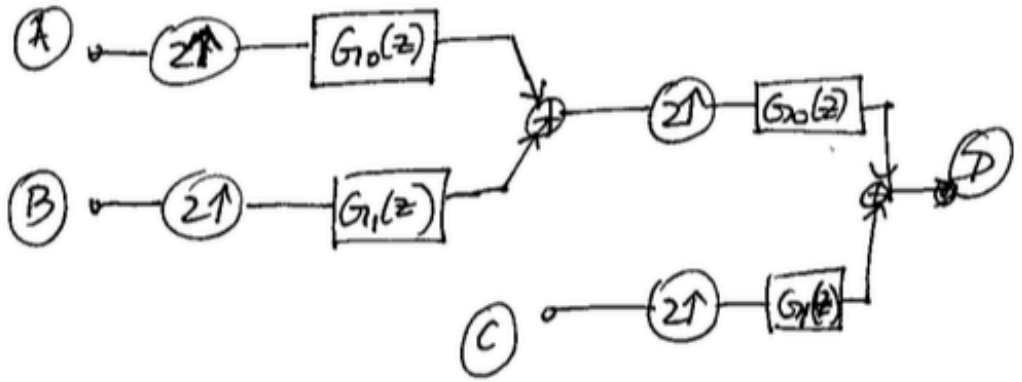
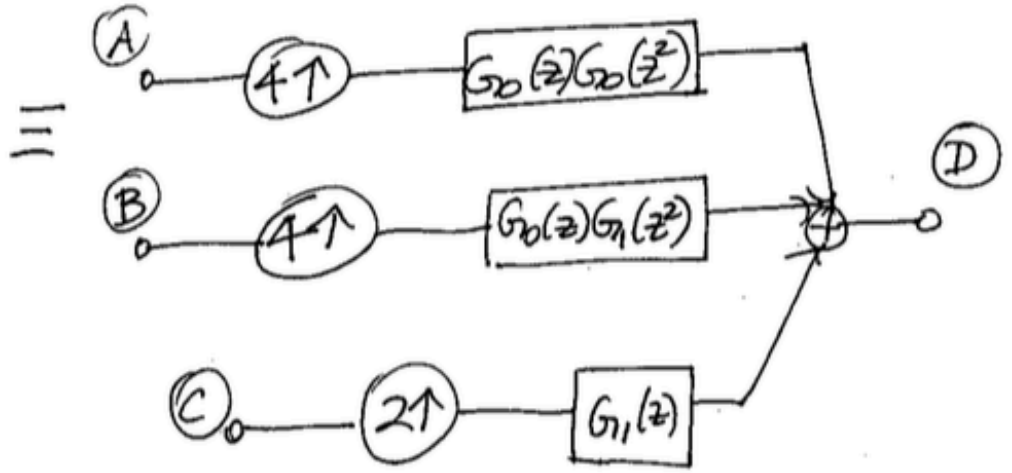


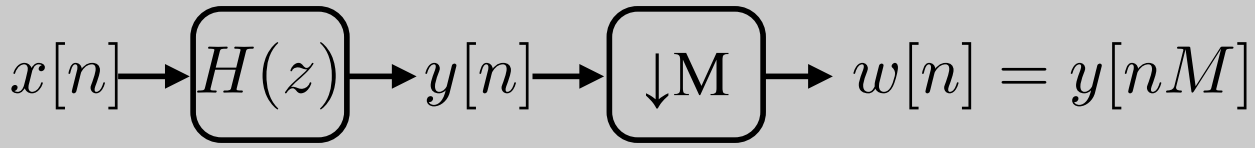
Ex.: Cascade of upsampling and filters

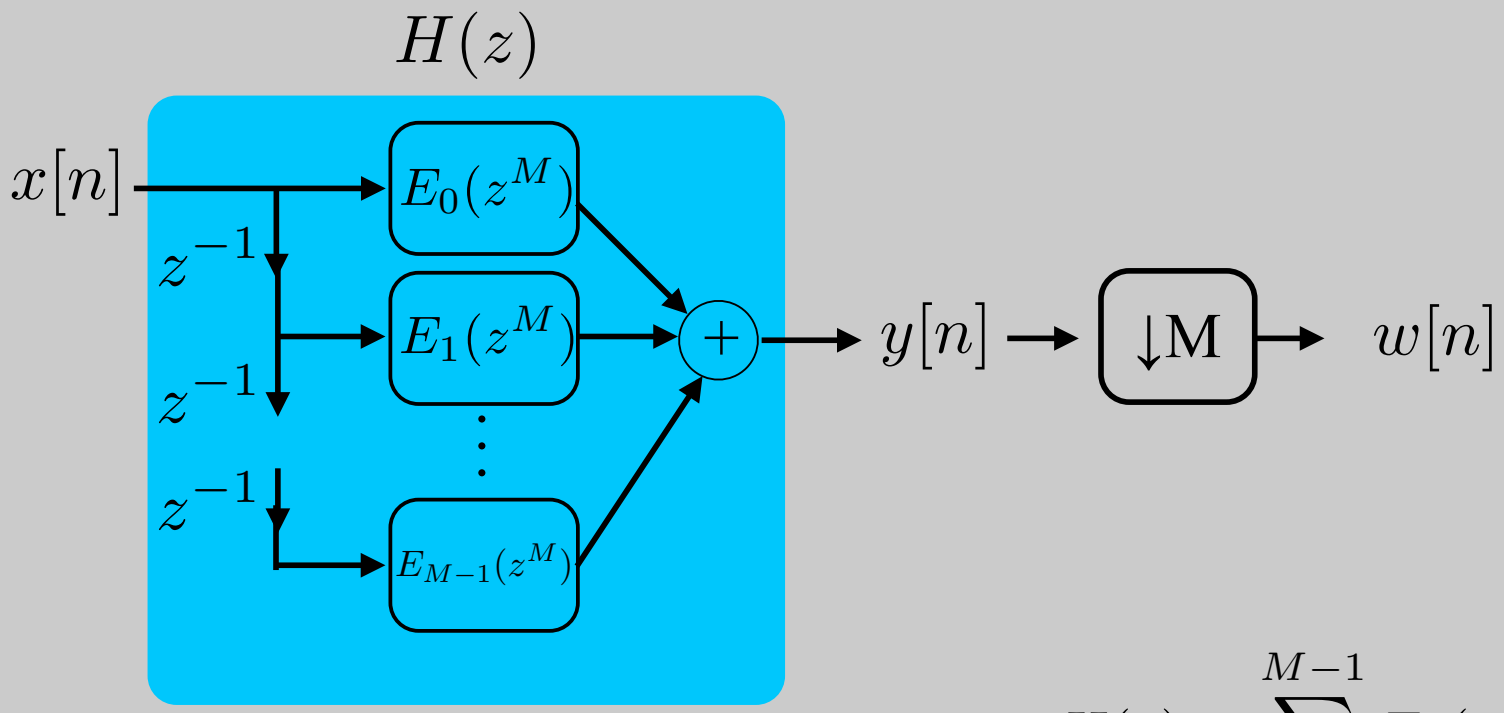






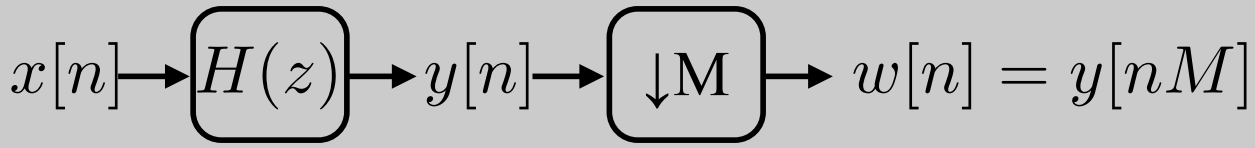
Polyphase Implementation of Decimation



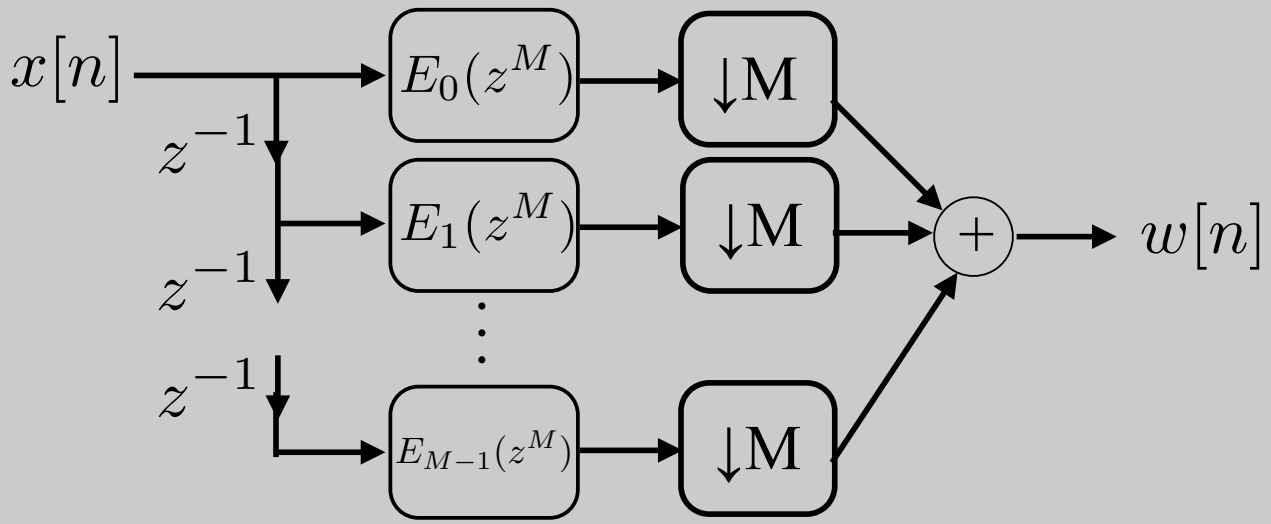


$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$

Polyphase Implementation of Decimation

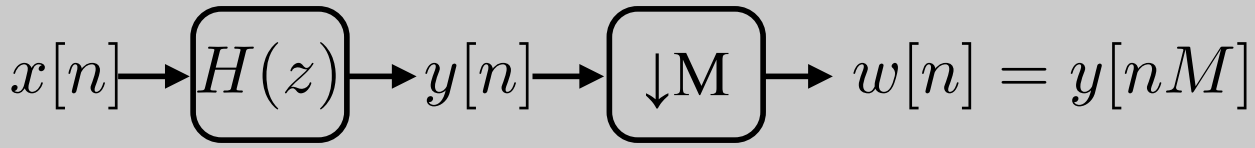


Interchange filter with decimation

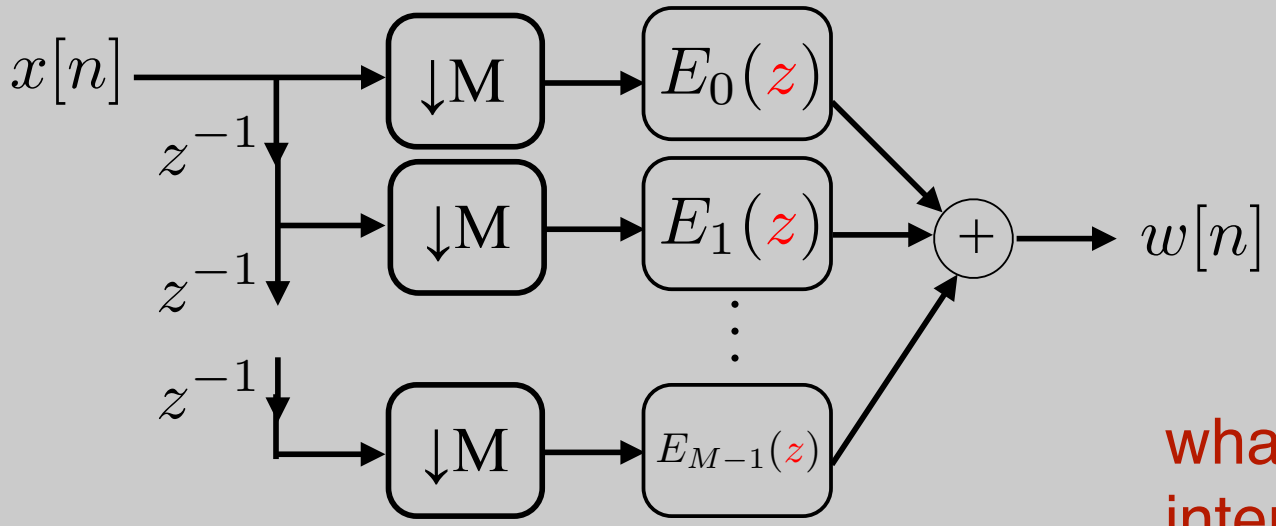


now, what can we do?

Polyphase Implementation of Decimation

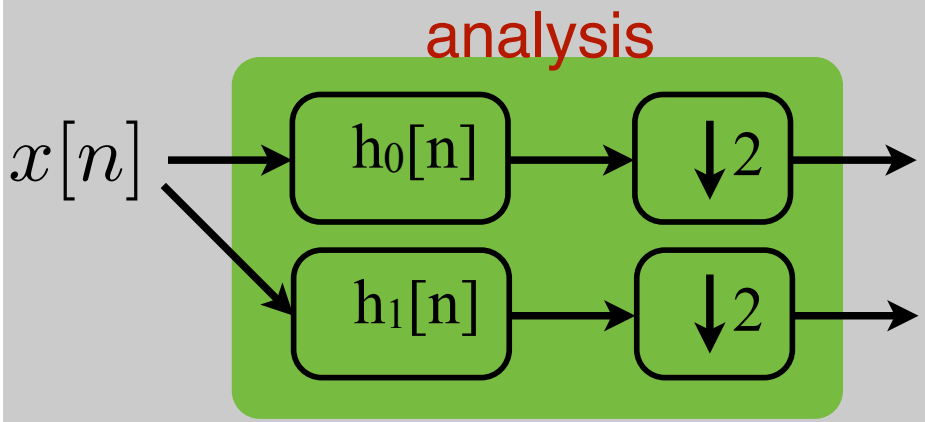


Interchange filter with decimation



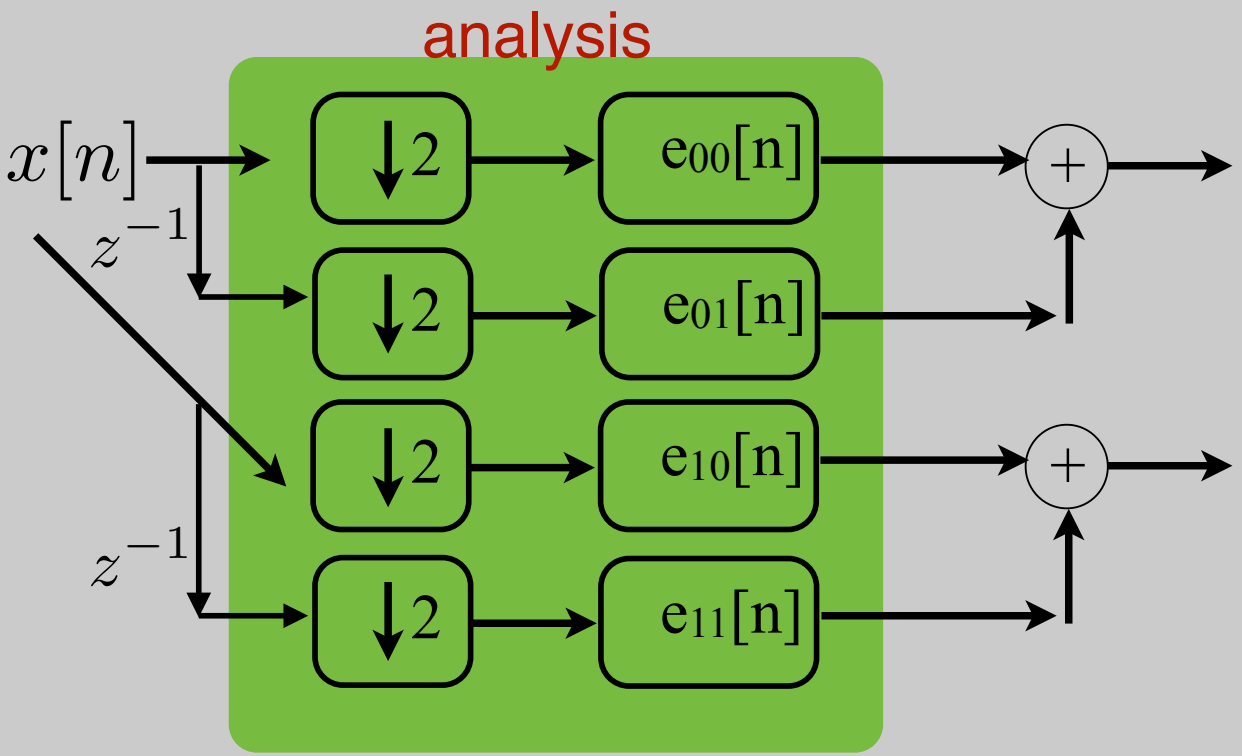
what about interpolation?

Computation:
Each Filter: N/M *(1/M) mult/unit time
Total: N/M mult/unit time



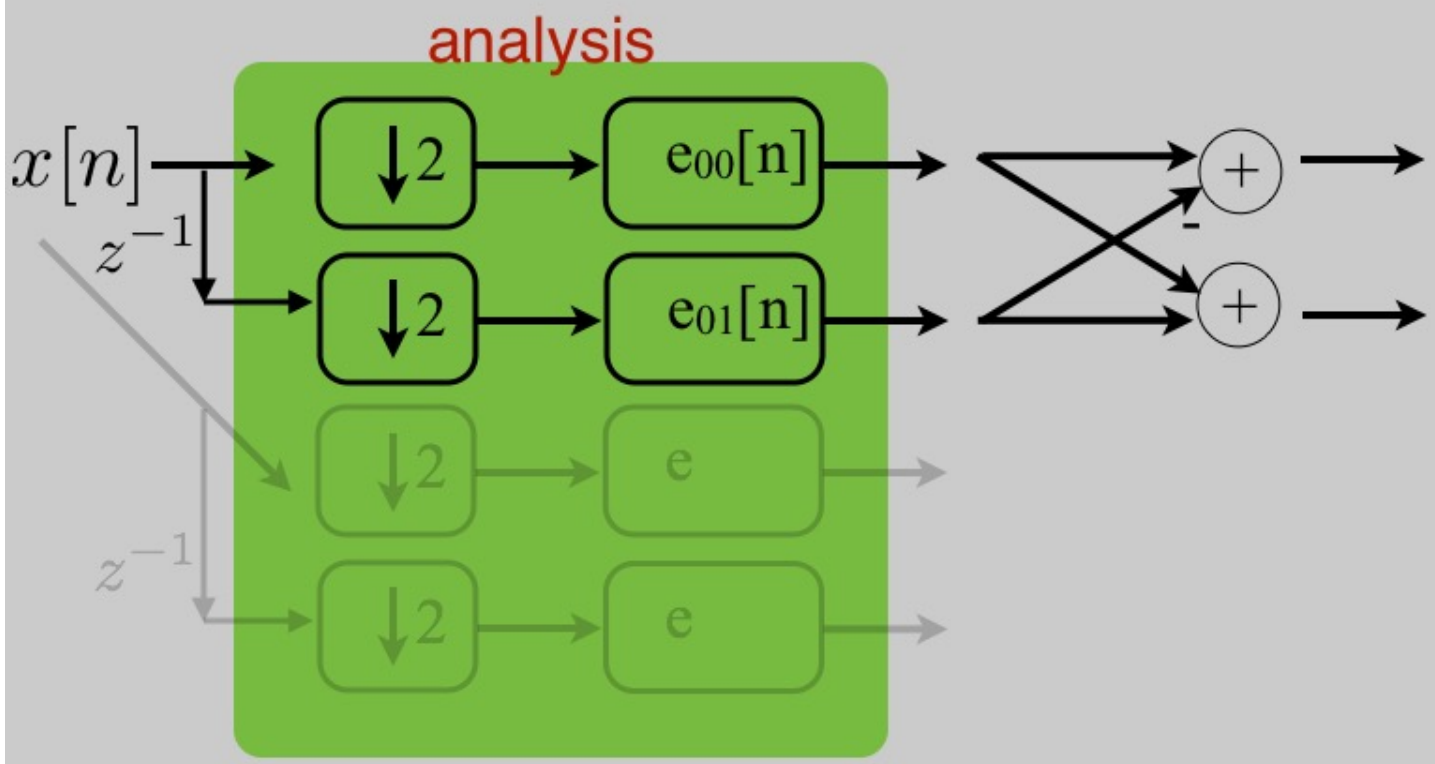
$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= h_1[2n] = e^{j2\pi n} h_0[2n] = e_{00}[n] \\
e_{11} &= h_1[2n+1] = e^{j2\pi n} e^{j\pi} h_0[2n+1] = -e_{01}[n]
\end{aligned}
$$

Polyphase Filter-Bank



$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{aligned}
$$

Polyphase Filter-Bank



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\begin{aligned}
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$$