

Fast Fourier Transform

Decimation in Time.

" Frequency.

"

Decimation in Time

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad 0 \leq k < N$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} + \sum_{\substack{n \rightarrow 0 \rightarrow N-1 \\ \text{odd}}} x(n) e^{-j2\pi nk/N}$$

$$X(k) = \sum_{\substack{n \text{ even} \\ 0 \rightarrow N-1}} x(n) e^{-j2\pi nk/N}$$

$$n = 2r \quad r: 0 \rightarrow \frac{N}{2} - 1$$

$$n = 2r + 1 \quad r: 0 \rightarrow \frac{N}{2} - 1$$

$$X(k) = \sum_{r=0}^{N/2-1} \underbrace{g(r)}_{x(2r)} e^{-j2\pi \frac{2r k}{N}} +$$

$$\sum_{r=0}^{N/2-1} \underbrace{g(r)}_{x(2r+1)} e^{-j2\pi \frac{(2r+1)k}{N}}$$

$$X(k) = \sum_{r=0}^{N/2-1} g(r) e^{-j2\pi \frac{r k}{N/2}} + e^{-j2\pi \frac{k}{N/2}}$$

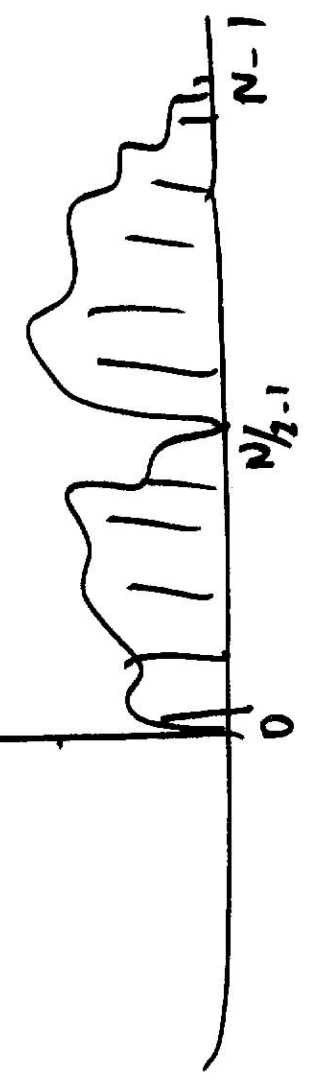
$\underbrace{\hspace{10em}}_{N/2 \text{ pt DFT of}}$

$N/2 \text{ pt seq. } g(r)$

Also DFS of $g(r)$

$k: 0 \rightarrow N-1$

$G(k)$



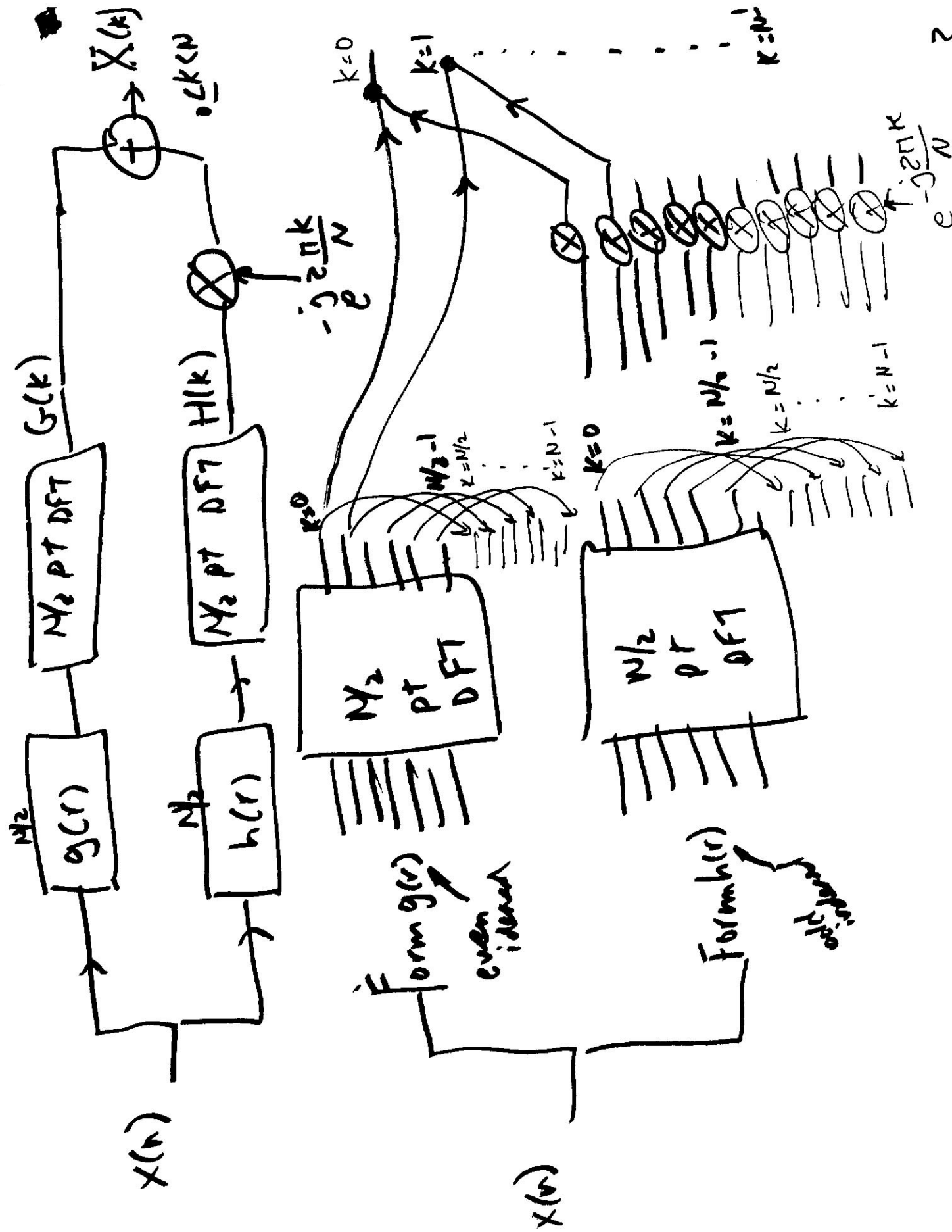
$$\sum_{r=0}^{N/2-1} \underbrace{h(r)}_{x(2r+1)} e^{-j2\pi \frac{(2r+1)k}{N}}$$

$$\sum_{r=0}^{N/2-1} h(r) e^{-j2\pi \frac{r k}{N/2}}$$

$\underbrace{\hspace{10em}}_{N/2 \text{ pt DFT}}$

of $N/2 \text{ pt seq.}$

$h(r)$
Also DFS of $h(r)$



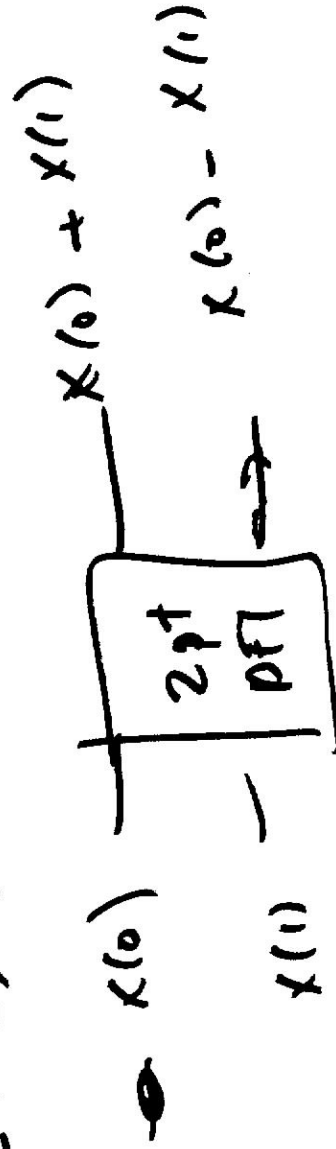
2pt DFT $x(n)$ $n=0,1$

$$\sum_{n=0}^1 x(n) e^{-j \frac{2\pi n k}{2}} = x(0) e^{-j \frac{2\pi n k}{2}} + x(1) e^{-j \frac{2\pi n k}{2}}$$

$$= x(0) + x(1) e^{-j \pi k}$$

$$k=0 \rightarrow e^{-j \pi k} = 1 \Rightarrow X(0) = x(0) + x(1)$$

$$k=1 \rightarrow e^{-j \pi k} = -1 \Rightarrow X(1) = x(0) - x(1)$$

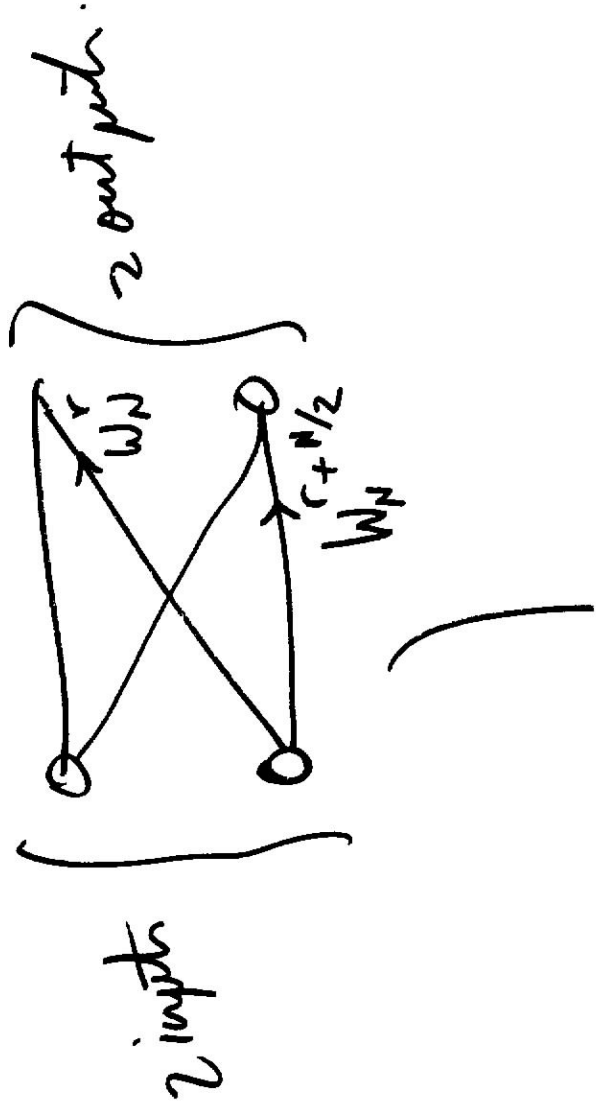


Show
pics
from 9.3-9.8

Define Twiddle factor

$$W_N^k \triangleq e^{-j \frac{2\pi k}{N}}$$

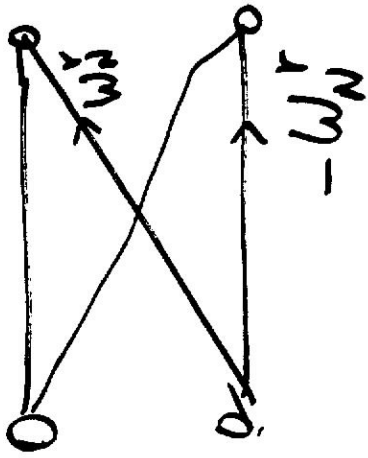
General form of a butterfly.



$$r + N/2 \quad -j \frac{2\pi}{N} (r + \frac{N}{2})$$

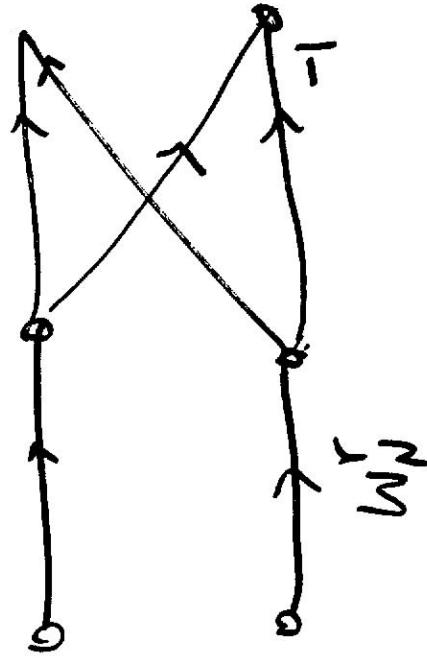
$$W_N \triangleq e^{-j \frac{2\pi r}{N}} = e^{-j \frac{2\pi r}{N}} = e^{-j \frac{2\pi r}{N}} = e^{-j \frac{2\pi r}{N}} = e^{-j \frac{2\pi r}{N}}$$

↓



2 mults
2 adds.

1 mult
2 adds



Show 9.10

Operation Count

- $\log_2 N$ stages.

- Each stage.

each butterfly

$N/2$ butterflies per stage.

Each butterfly has 2 adds & 1 mult.

N inputs \rightarrow N outputs
 2 inputs \rightarrow 2 outputs.

- Total

$$\left(\begin{array}{l} \frac{N}{2} \times 2 \log_2 N. \text{ adds} \approx \frac{N \log_2 N}{2} \\ \frac{N}{2} \times 1 \times \log_2 N \text{ mult.} \frac{N \log_2 N}{2} \end{array} \right)$$

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Bit Reversing
input/output order

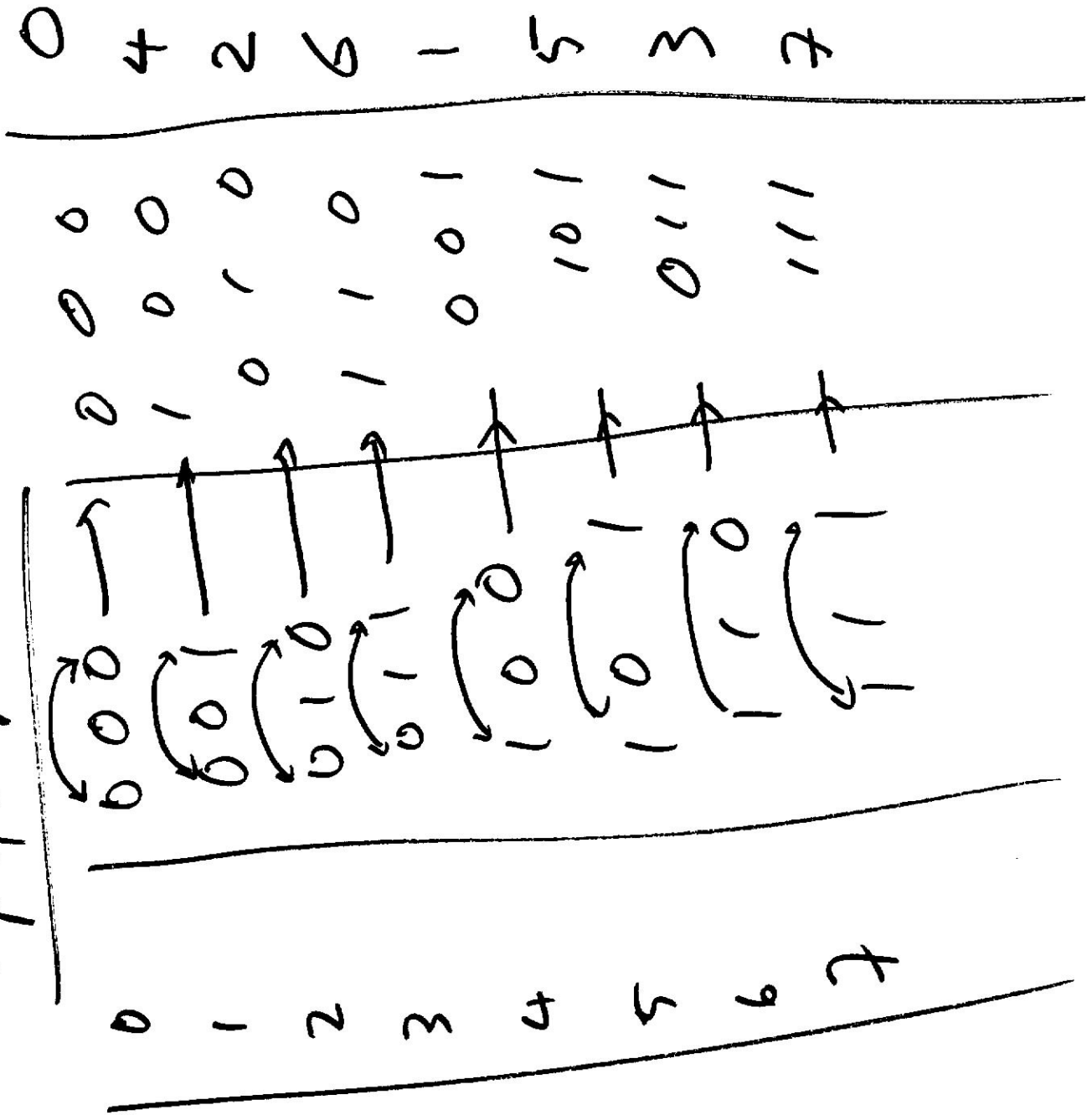


Fig 9.10

- x(0)
- x(4)
- x(2)
- x(6)
- x(1)
- x(5)
- x(3)
- x(7)

Decimation in Frequency

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$$

Divide computation 2 parts: $k=2r$ even indices of k
 $k=2r+1$ odd indices of k .

① k even $k=2r$

$$X(2r) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nr/N}$$

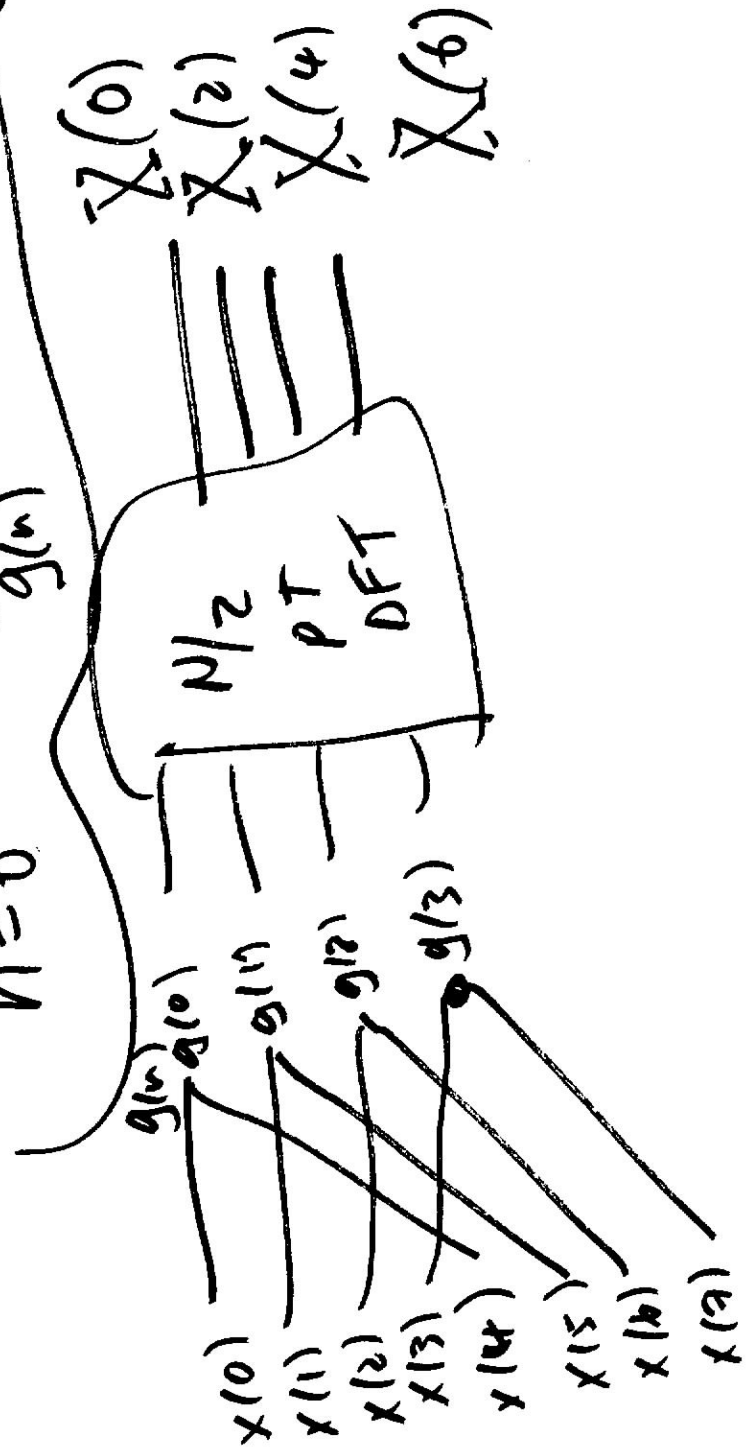
$$0 \leq r < N/2$$

$$X(2r) = \sum_{n=0}^{N/2-1} x(n) e^{-j2\pi nr/N} + \sum_{n=N/2}^{N-1} x(n) e^{-j2\pi nr/N}$$

$$= \sum_{m=0}^{N/2-1} x(m) e^{-j2\pi r m/N} + \sum_{m=0}^{N/2-1} x(m + N/2) e^{-j2\pi r (m + N/2)/N}$$

$$X(2r) = \sum_{n=0}^{N/2-1} x(n) e^{-j2\pi r n / N/2} + \sum_{n=0}^{N/2-1} x(n + \frac{N}{2}) e^{-j2\pi r n / N/2}$$

$$X(2r) = \sum_{n=0}^{N/2-1} \underbrace{\left[x(n) + x(n + \frac{N}{2}) \right]}_{g(n)} e^{-j2\pi r n / N/2} \quad 0 \leq r < N/2$$



How about $k = 2r+1$, i.e. odd indices of k .

$$X(2r+1) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi n (2r+1)}{N}} \quad 0 \leq r < \frac{N}{2}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n) e^{-j \frac{2\pi n (2r+1)}{N}} + \sum_{n=N/2}^{N-1} x(n) e^{-j \frac{2\pi n (2r+1)}{N}}$$

$$X(2r+1) = \sum_{m=0}^{N/2-1} x\left(m + \frac{N}{2}\right) e^{-j \frac{2\pi}{N} \left(m + \frac{N}{2}\right) (2r+1)}$$

$$e^{-j \frac{2\pi}{N} \frac{N}{2} (2r+1)} \sum_{m=0}^{N/2-1} x\left(m + \frac{N}{2}\right) e^{-j \frac{2\pi m}{N} (2r+1)}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n) e^{-j2\pi n(2r+1)/N} - \sum_{n=0}^{N/2-1} x(n+\frac{N}{2}) e^{-j2\pi n(2r+1)/N}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} [x(n) - x(n+\frac{N}{2})] e^{-j2\pi n(2r+1)/N}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \underbrace{(x(n) - x(n+\frac{N}{2})) e^{-j2\pi n(2r+1)/N}}_{h(n)}$$

$N/2$ pt DFT $h(n)$

