Digital Signal Processing

Lecture 10C+D
Transform Analysis of LTI Systems

* Beautiful handwritten figures by Prof. Murat Arcak
Admin

- Midterm Friday
  - 2 hours
  - open everything
  - Review during class Wednesday. Post requested topics to @143 on Piazza

- Lab checkins this week

- Lab 4 prelab out today, due Friday
  - Bring pi and speaker. We want to hear sub-bands!

- Homework 8 out Wednesday
All pass systems

What is the magnitude response of

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$
Magnitude response of all-pass

\[ |H(e^{jw})|^2 = \frac{|e^{-jw} - a^*|^2}{|1 - ae^{-jw}|^2} = \frac{|e^{-jw}(1 - a^*e^{jw})|^2}{|1 - ae^{-jw}|^2} \]

\[ = \frac{|1 - (ae^{-jw})^*|^2}{|1 - ae^{-jw}|^2} = \frac{(1 - (ae^{-jw})^*)(1 - (ae^{-jw})^*)^*}{(1 - ae^{-jw})(1 - ae^{-jw})^*} \]

\[ = \frac{(1 - (ae^{-jw})^*)(1 - ae^{-jw})}{(1 - ae^{-jw})(1 - (ae^{-jw})^*)} = 1 \quad \forall w \]
A general all-pass system (with real-valued impulse response)

\[
H_{ap}(z) = \prod_{k=1}^{M_C^{\text{Re}}} \frac{z^{-1}}{1 - d_k z^{-1}} \cdot \prod_{k=1}^{M_C^{\text{Comp}}} \frac{z^{-1} - e_{k}^*}{1 - e_{k} z^{-1} - e_{k}^* z^{-1}}
\]

\(d_k\): real poles
\(e_k\): complex poles paired w/ conjugate \(e_k^*\)

\[
|H_{ap}(e^{j\omega})| = 1
\]

Example
$|H(e^{j\omega})|$  

**All values = 1 plotted in black**

**all pass magnitude, $a = 0.70$**

**all pass magnitude, $a = 0.90$**
phase response of an all-pass:

\[
\arg\left(\frac{e^{-j\omega} - re^{j\theta}}{1 - re^{j\theta} e^{-j\omega}}\right) = \arg\left(\frac{e^{-j\omega} (1 - re^{-j\theta} e^{j\omega})}{1 - re^{j\theta} e^{-j\omega}}\right) = \\
\arg\left(e^{-j\omega}\right) - 2\arg\left(1 - re^{j\theta} e^{-j\omega}\right)
\]

\[
g\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta} e^{-j\omega}}\right) = 1 - 2g\left(1 - re^{j\theta} e^{-j\omega}\right)
\]
Example:

\[ \text{Figure 5.20} \]

\begin{itemize}
\item (a) \text{Radians} \\
\item (b) \text{Samples}
\end{itemize}

\text{can be used to compensate phase distortion.}
Claim: for a stable op system $H_o(p)$:

1. $\text{grd} \left[ H_o(e^{j\omega}) \right] > 0$
2. $\text{arg} \left[ H_o(e^{j\omega}) \right] < 0$

Delay positive $\rightarrow$ causal
Phase negative $\rightarrow$ phase lag.

proof in book.
Root flipping (added post lecture)

• Given a system with magnitude $C(e^{jw}) = |H(e^{jw})|^2$
• Let $H(z) = (1 - az^{-1})$ (zero at a), where a is complex
• $|H(e^{jw})|^2 = (1 - ae^{-jw})(1 - ae^{-jw})^*$

\[
|a|^2 \cdot \frac{1}{|a|^2} (1 - ae^{-jw})(1 - a^*e^{jw})
= |a|^2 \cdot \frac{1}{aa^*} (1 + aa^* - ae^{-jw} - a^*e^{jw})
= |a|^2 \cdot \left( \frac{1}{aa^*} + 1 - \frac{e^{-jw}}{a^*} - \frac{e^{jw}}{a} \right)
= |a|^2 \left( 1 - \frac{e^{-jw}}{a^*} \right) \left( 1 - \frac{e^{jw}}{a} \right)
= |a|^2 \left( 1 - \frac{e^{-jw}}{a^*} \right) \left( 1 - \frac{e^{-jw}}{a^*} \right)^*
= |a|^2 |H(e^{jw})|^2
\]

• So if the zero is moved from $a$ to $1/a^*$, the magnitude of the transfer function (on the unit circle) is the same up to a constant.
• Given a magnitude, the locations of the zeros are ambiguous!
Minimum-Phase Systems

Definition: a stable and causal system $H(z)$

Poles are inside the unit circle

The inverse of $\frac{1}{H(z)}$ is also stable & causal
Equivalently: construct minimum phase by moving all zeros inside unit circle
Example

\[ H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{3}z^{-1}} \]

**Set:**

\[ H_{op} = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}} \]

\[ H_{min}(z) = \frac{H(z)}{H_{op}(z)} \]

\[ H_{min}(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{3}z^{-1}} \cdot \frac{z^{-1} - \frac{1}{3}z^{-1}}{z^{-1} - \frac{1}{3}} = -3 \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}} \]
why m.p. property important?

\[ s[n] \rightarrow \text{Distortion} \rightarrow H_d(z) \rightarrow \text{Compensation} \rightarrow H_c(z) \rightarrow s_c[n] \]

for ex. communication chan.

If \( H_d(z) \) is minimum phase, design

\[ H_c(z) = \frac{1}{H_d(z)} \quad \text{(stable?)} \]

If not m.p., decompose: \( H_d(z) = H_d, m_p(z) \cdot H_d, m_{ap}(z) \)

\[ H_c(z) = \frac{1}{H_d, \text{min}(z)} \Rightarrow H_d H_c = H_d, m_{ap}(z) \]

only compensate for mag.
Why call “minimum phase”? Different systems can have same mag. response.

$H_4(z)$ min phase?

Recall:

$H(z) = H_{\text{min}}(z) \cdot H_{\text{op}}(z)$

$H_2(z)$ (max phase)

$H_3(z)$

$H_4(z)$

$H_2 = H_4 \cdot H_{\text{op},2}$

$H_3 = H_4 \cdot H_{\text{op},3}$

$H_4 = H_4 \cdot H_{\text{op},4}$
of all, $H_u(t)$ has minimum phase by (12)

because:

$$\arg[H(e^{jw})] = \arg[H_{\text{min}}(e^{jw})] + \arg[H_{\text{ap}}(e^{jw})] \leq 0$$

other properties:

minimum group delay:

$$\text{grd}[H(e^{jw})] = \text{grd}[H_{\text{min}}] + \text{grd}[H_{\text{ap}}]$$

minimum energy delay:

Problem 5.72
Minimum-phase VS Linear Phase

linear phase

minimum phase

Graphs showing the comparison between linear phase and minimum phase signals.