EE123
Digital Signal Processing

Lecture 10C+D
Transform Analysis of LTI Systems

* Beautiful handwritten figures by Prof. Murat Arcak

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Admin

- Midterm Friday
  - 2 hours
  - open everything
  - Review during class Wednesday. Post requested topics to @143 on Piazza
- Lab checkins this week
- Lab 4 prelab out today, due Friday
  - Bring pi and speaker. We want to hear sub-bands!
- Homework 8 out Wednesday

All pass systems

What is the magnitude response of

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

Magnitude response of all-pass

$$\left|H\left(e^{jw}\right)\right|^2 = \frac{\left|e^{-jw} - a^*\right|^2}{\left|1 - ae^{-\mathrm{jw}}\right|^2} = \frac{\left|e^{-jw}\left(1 - a^* e^{jw}\right)\right|^2}{\left|1 - ae^{-\mathrm{jw}}\right|^2}$$

$$= \frac{\left|1 - \left(ae^{-jw}\right)^*\right|^2}{\left|1 - ae^{-\mathrm{jw}}\right|^2} = \frac{\left(1 - \left(ae^{-jw}\right)^*\right)\left(1 - \left(ae^{-jw}\right)^*\right)^*}{\left(1 - ae^{-\mathrm{jw}}\right)\left(1 - ae^{-\mathrm{jw}}\right)^*}$$

$$= \frac{\left(1 - \left(ae^{-jw}\right)^*\right)\left(1 - ae^{-\mathrm{jw}}\right)}{\left(1 - ae^{-\mathrm{jw}}\right)\left(\left(1 - (ae^{-\mathrm{jw}})^*\right)\right)} = 1 \quad \forall w$$

A general all-pass system (with real-valued impulse response)

$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \cdot \prod_{k=1}^{M_c} \frac{z^{-1} - e_k^*}{1 - e_k z^{-1}} \cdot \frac{z^{-1} - e_k}{1 - e_k^* z^{-1}}$$

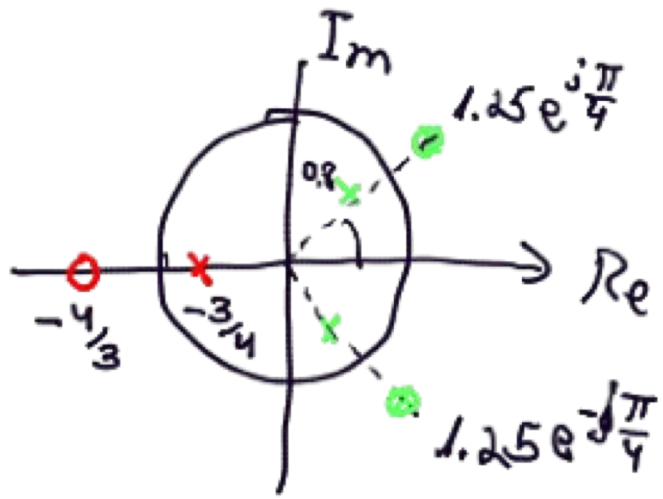
Real Complex

d_k: real Poles

e_k: complex poles paired w/ conjugate e_k^*

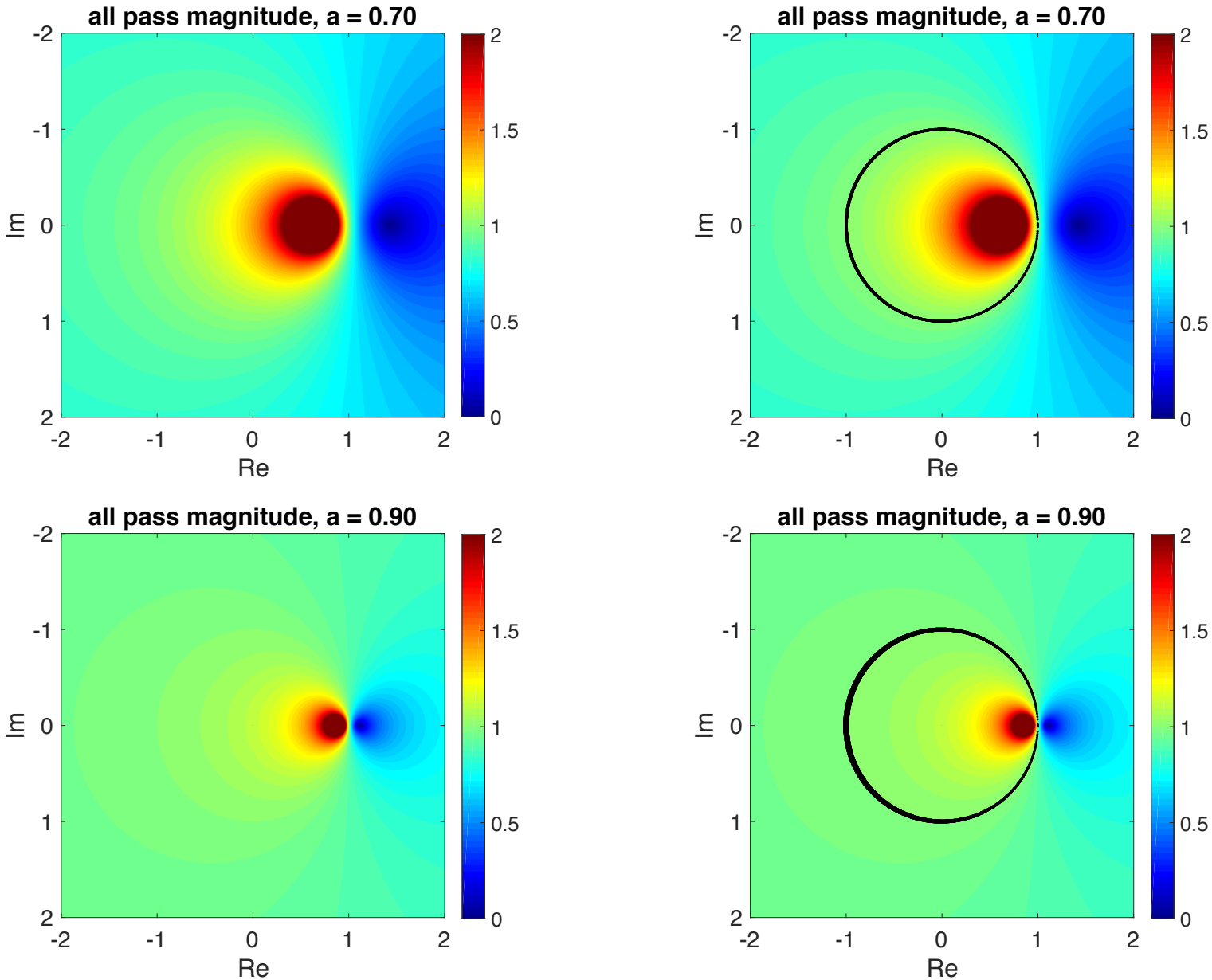
$$|H_{ap}(e^{j\omega})| \equiv 1$$

Example



$\left|H\left(e^{jw}\right)\right|$

All values = 1 plotted in black



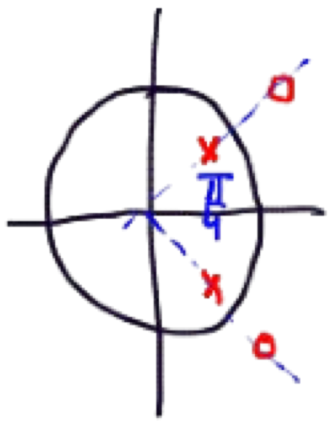
phase response of an all-pass:

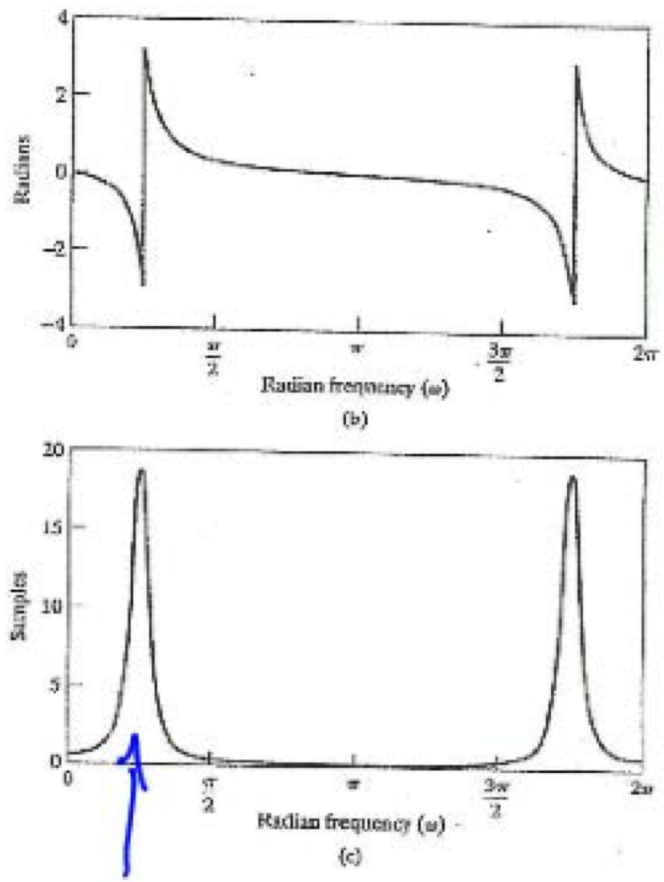
$$\arg\left[\frac{e^{-j\omega} - \overbrace{re^{-j\theta}}^{a^*}}{1 - \underbrace{re^{j\theta}}_{a} e^{-j\omega}}\right] = \arg\left[\frac{e^{-j\omega}(1 - re^{-j\theta}e^{+j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right] =$$

$$= \underbrace{\arg\left[e^{-j\omega}\right]}_{-\omega} - 2\arg\left[1 - re^{j\theta}e^{-j\omega}\right]$$

$$\mathrm{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = 1 - 2\,\mathrm{grd}\left[1 - re^{j\theta}e^{-j\omega}\right]$$

Example: <figure 5.20>





can be used to compensate phase distortion.

<u>Claim</u>: for a stable ap system $H_{ap}(z)$:

(i) $\text{grd}\left[H_{ap}(e^{j\omega})\right] > 0$

(ii) $\arg\left[H_{ap}(e^{j\omega})\right] \leq 0$

Delay positive → causal
phase negative → phase lag.

proof in book.

Root flipping (added post lecture)

- Given a system with magnitude $C(e^{jw}) = \left|H(e^{jw})\right|^2$
- Let $H(z) = (1 - az^{-1})$ (zero at a), where a is complex
- $\left|H(e^{jw})\right|^2 = (1 - ae^{-jw})(1 - ae^{-jw})^*$

$$
\begin{aligned}
&|a|^2 \cdot \frac{1}{|a|^2}(1 - ae^{-jw})(1 - a^* e^{jw}) \\
&= |a|^2 \cdot \frac{1}{aa^*}(1 + aa^* - ae^{-jw} - a^* e^{jw}) \\
&= |a|^2 \cdot \left(\frac{1}{aa^*} + 1 - \frac{e^{-jw}}{a^*} - \frac{e^{jw}}{a}\right) \\
&= |a|^2 \left(1 - \frac{e^{-jw}}{a^*}\right)\left(1 - \frac{e^{jw}}{a}\right) \\
&= |a|^2 \left(1 - \frac{e^{-jw}}{a^*}\right)\left(1 - \frac{e^{-jw}}{a^*}\right)^* \\
&= |a|^2 \left|H(e^{jw})\right|^2
\end{aligned}
$$

- So if the zero is moved from a to $1/a^*$, the magnitude of the transfer function (on the unit circle) is the same up to a constant.
- Given a magnitude, the locations of the zeros are ambiguous!

Minimum-Phase Systems

Definition: a $\underbrace{\text{stable and causal}}_{\text{Poles are inside the unit circle}}$ system $H(z)$

whose inverse $\frac{1}{H(z)}$ is also stable & causal

AP-Min-Phase decomposition:

stable, causal system can be decomposed to:

$$H(z) = \underbrace{H_{min}(z)}_{\text{min phase}} \cdot \underbrace{H_{ap}(z)}_{\text{all pass}}$$

Approach: ① first construct H_{ap} with all zeros outside unit circle

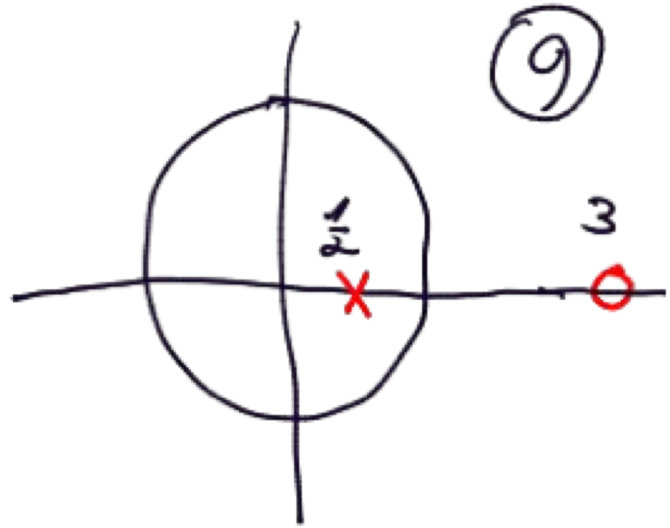
② compute

$$H_{min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Equivalently: construct minimum phase by moving all zeros inside unit circle

Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$



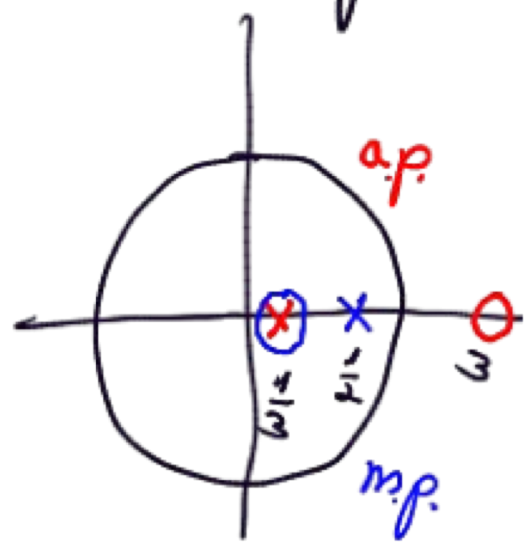
Set:

$$H_{ap} = \frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$$

$$H_{min}(z) = \frac{H(z)}{H_{ap}(z)}$$

$$H_{min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1-\frac{1}{3}z^{-1}}{z^{-1}-\frac{1}{3}} = -3\,\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$-3 =$



why m.p. property important?

for ex. communication chan.

If $H_d(z)$ is minimum phase, design

$$H_c(z) = \frac{1}{H_d(z)} \quad \text{(stable!)}$$

If not m.p., decompose: $H_d(z) = H_{d,mp}(z) \cdot H_{d,ap}(z)$

$$H_c(z) = \frac{1}{H_{d,min}(z)} \Rightarrow H_d H_c = H_{d,ap}(z)$$

only compensate for mag.

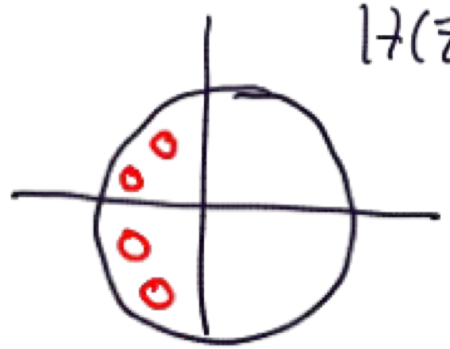
Why called "minimum phase"?

Different systems can have same mag. response.

Recall:

$$H(z) = \underbrace{H_{min}(z)}_{\text{min phase}} \cdot \underbrace{H_{ap}(z)}_{\text{all pass}}$$

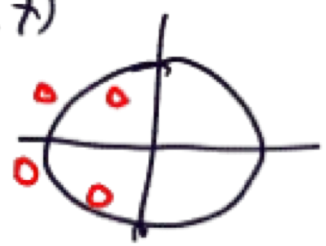
$H_1(z)$ min phase:



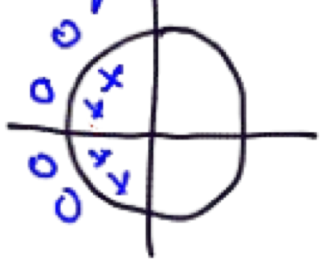
$H_2(z)$ (max phase)

$H_3(z)$

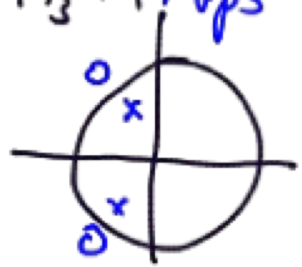
$H_4(z)$



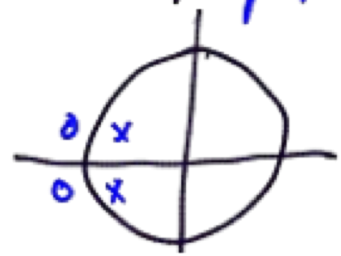
$H_2 = H_1 H_{ap,2}$



$H_3 = H_1 H_{ap3}$



$H_4 = H_1 H_{ap4}$



of all, $H_1(z)$ has minimum phase by (12)

because:

$$\arg[H(e^{jw})] = \arg[H_{min}(e^{jw})] + \underbrace{\arg[H_{ap}(e^{jw})]}_{\leq 0}$$

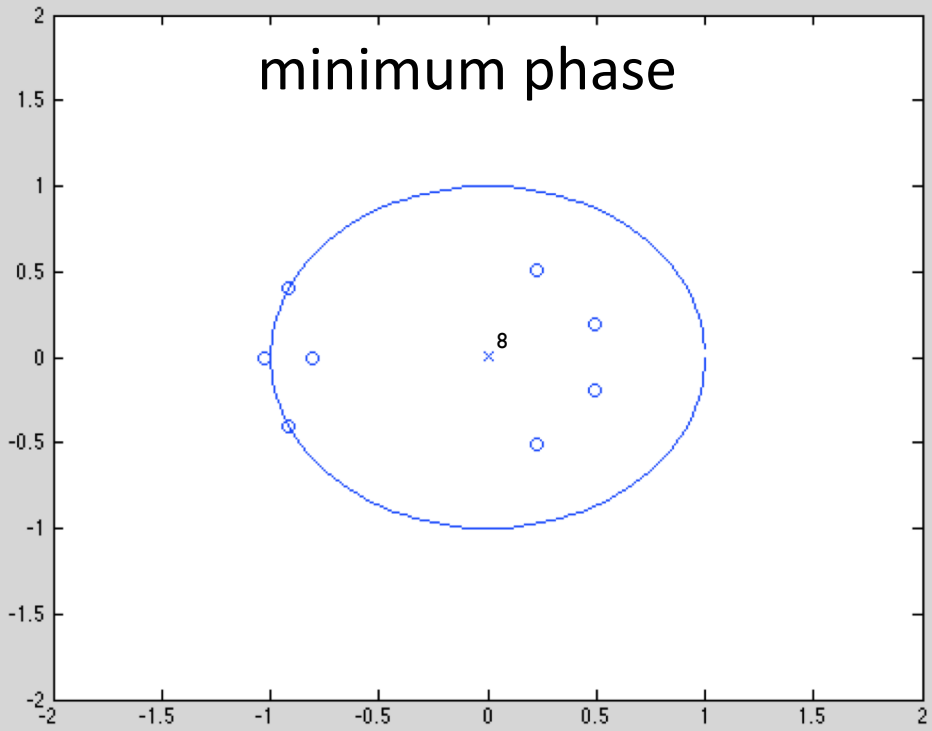
other properties:

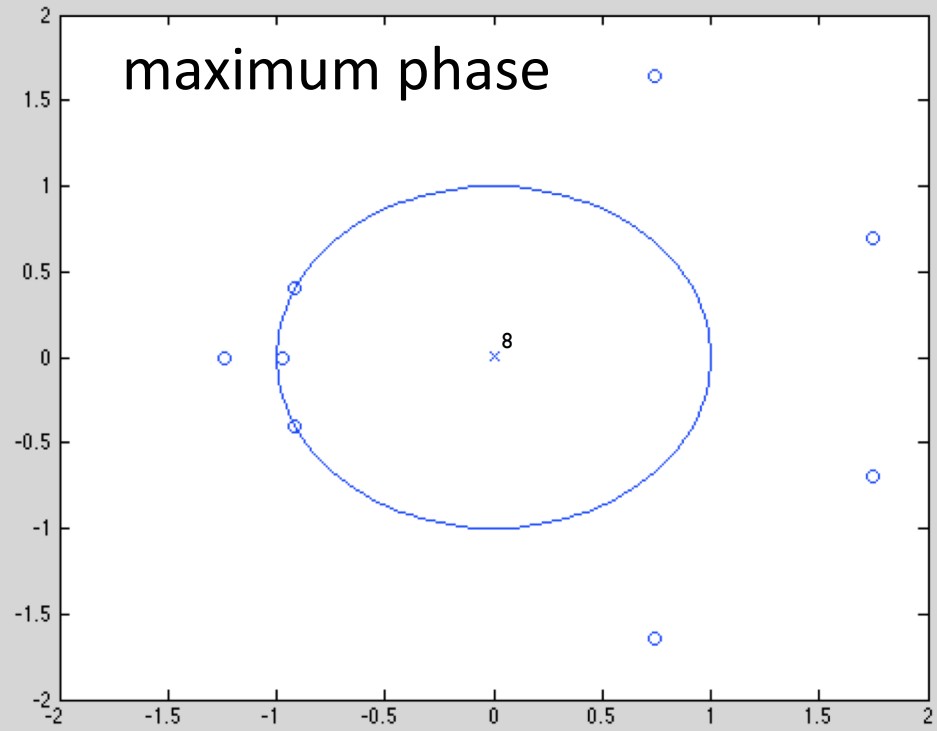
<u>minimum group delay</u>:

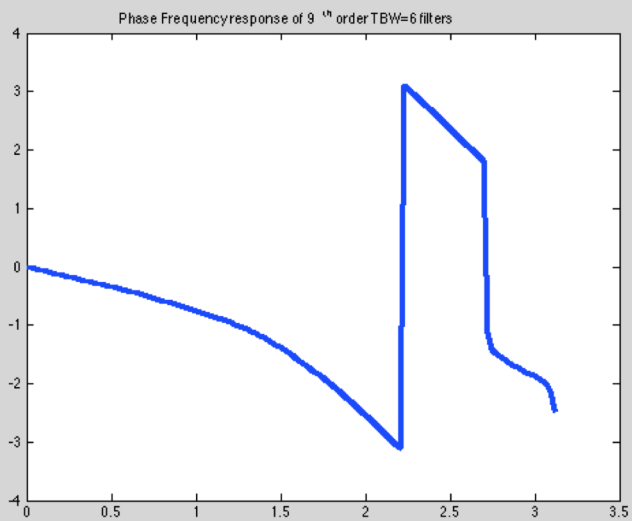
$$grd[H(e^{j\omega})] = grd[H_{min}] + \underbrace{grd[H_{ap}]}_{\geq 0}$$

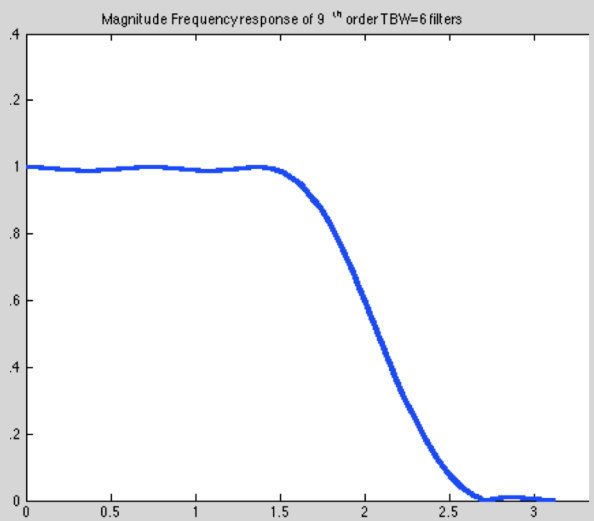
<u>minimum energy delay</u>:

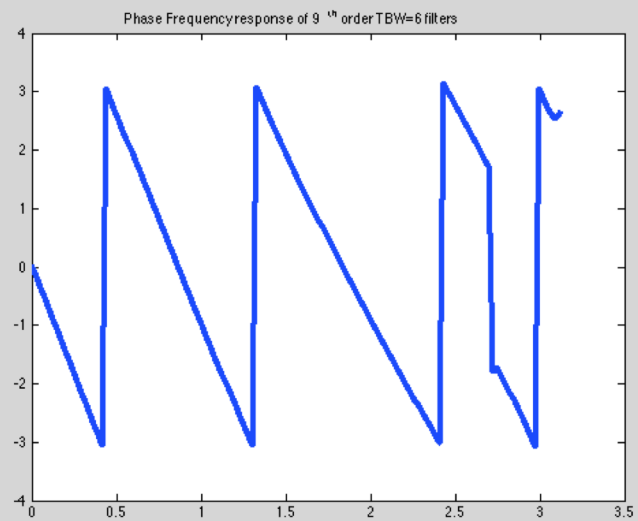
Problem 5.72


minimum phase
8


maximum phase
8

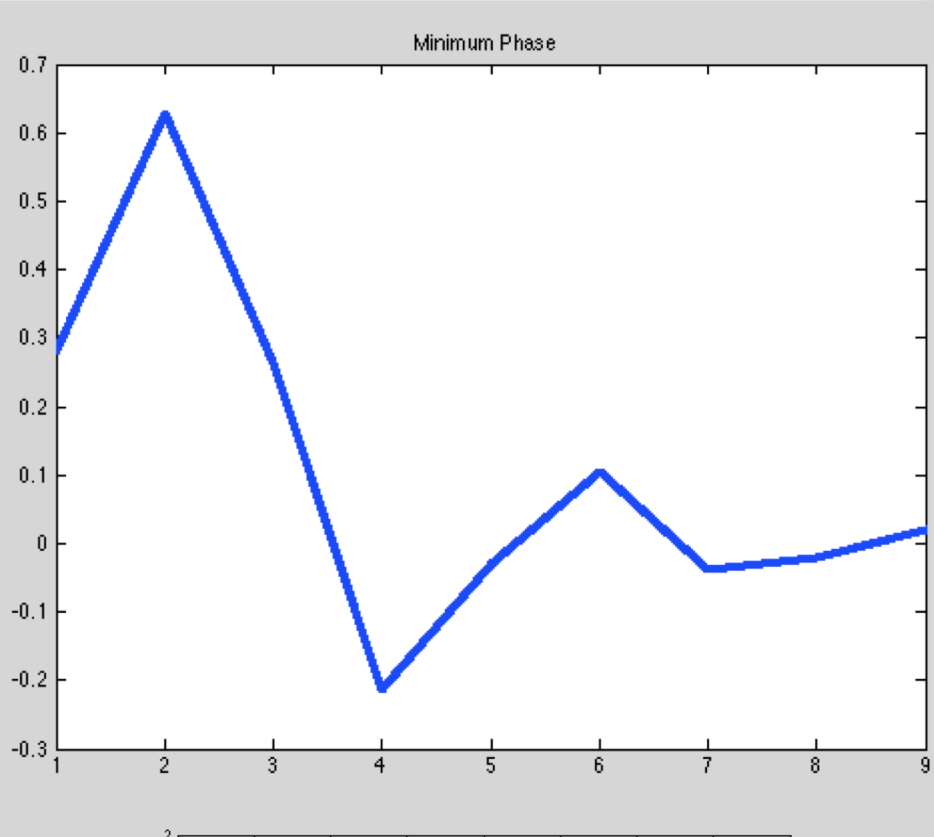

Phase Frequencyresponse of 9 th order TBW=6 filters

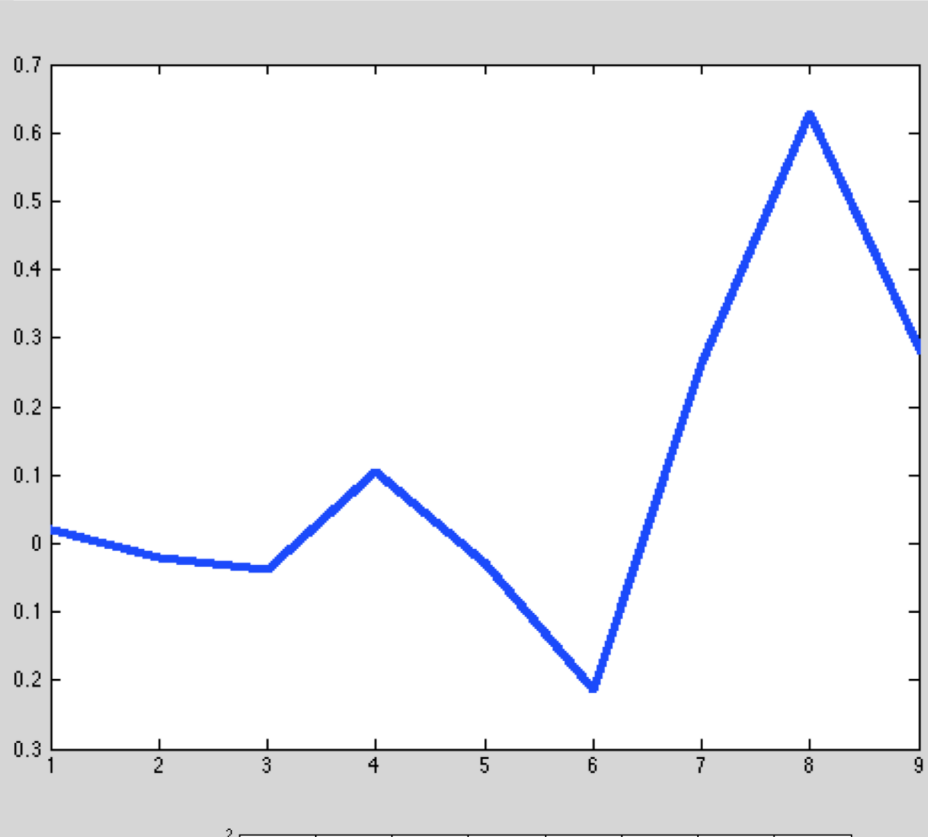

Magnitude Frequencyresponse of 9 th order TBW=6 filters

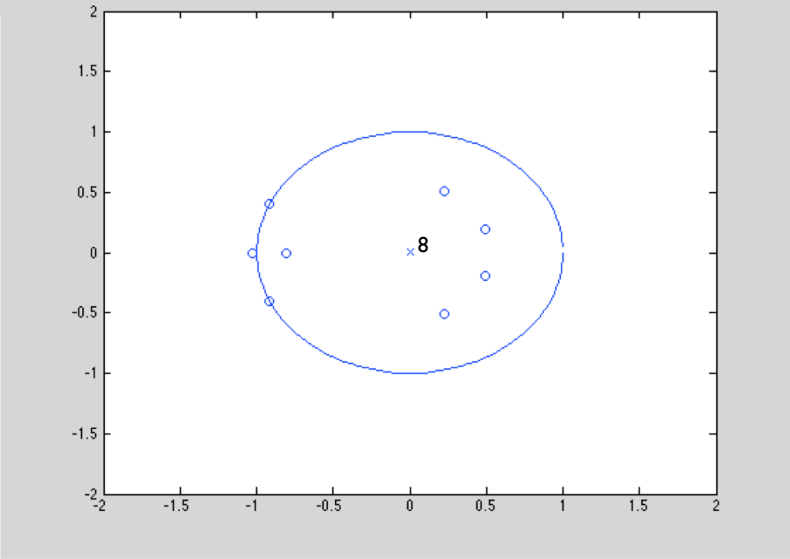

Phase Frequencyresponse of 9 th order TBW=6 filters

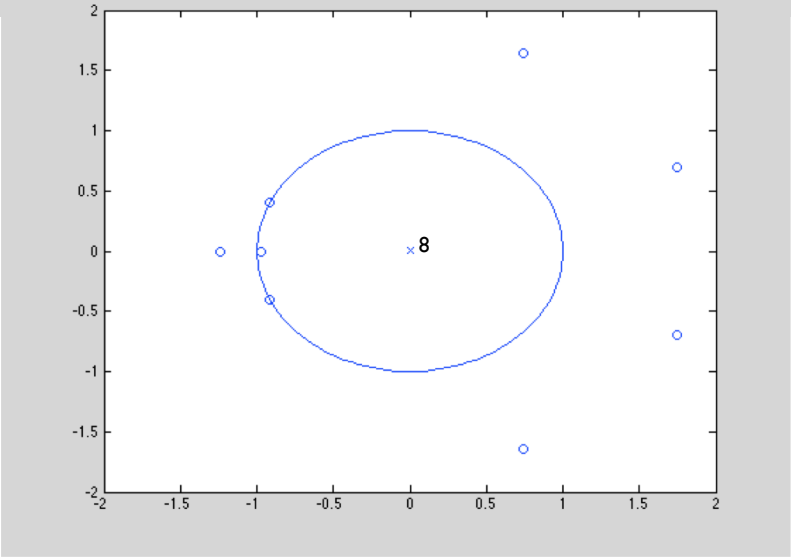
minimum phase

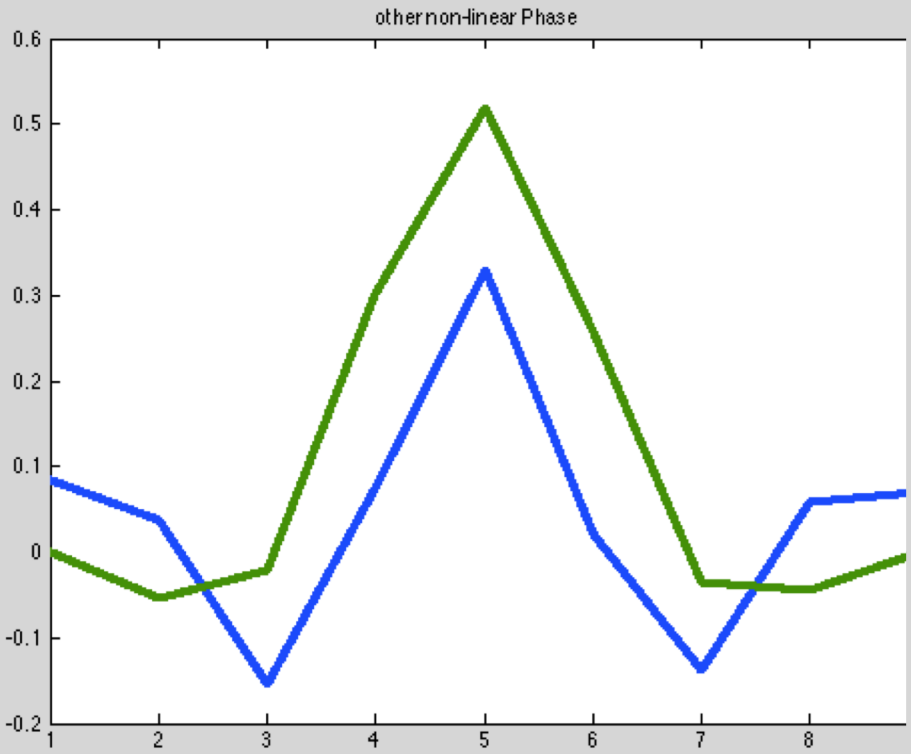
maximum phase

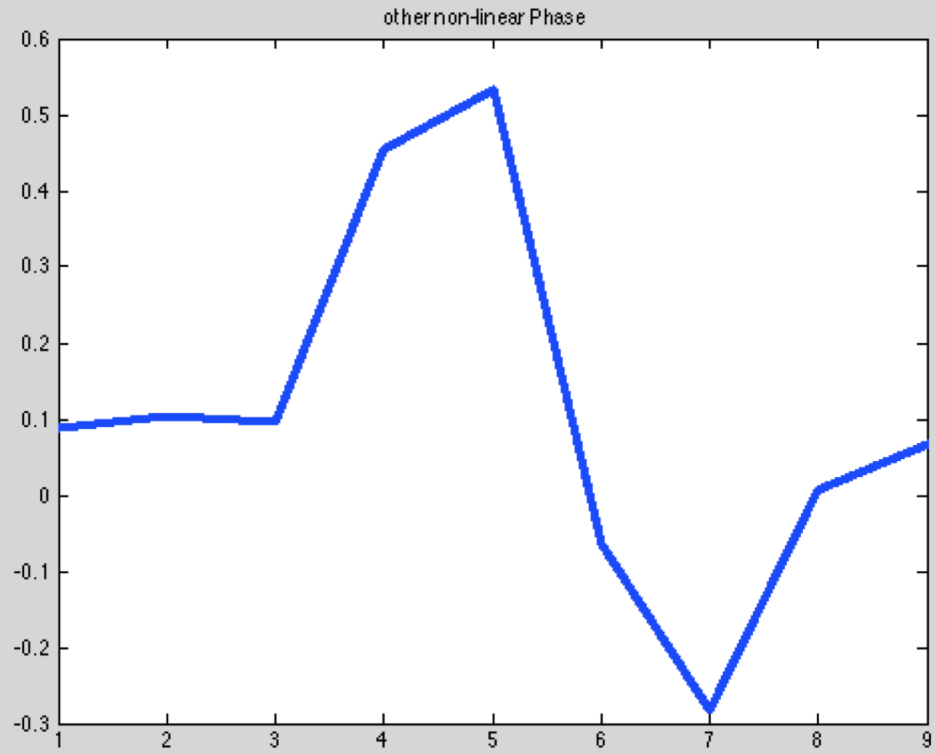









other non-linear Phase
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
1
2
3
4
5
6
7
8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0.6
0.5
0.4
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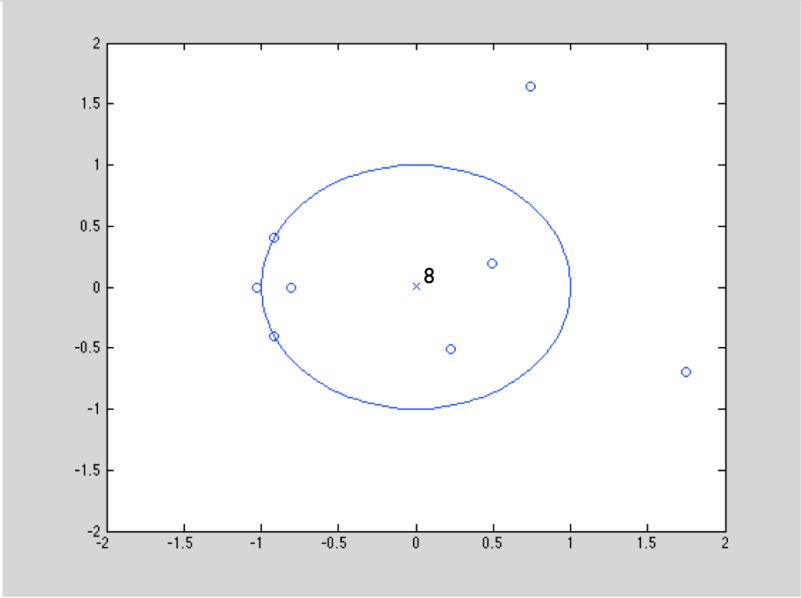
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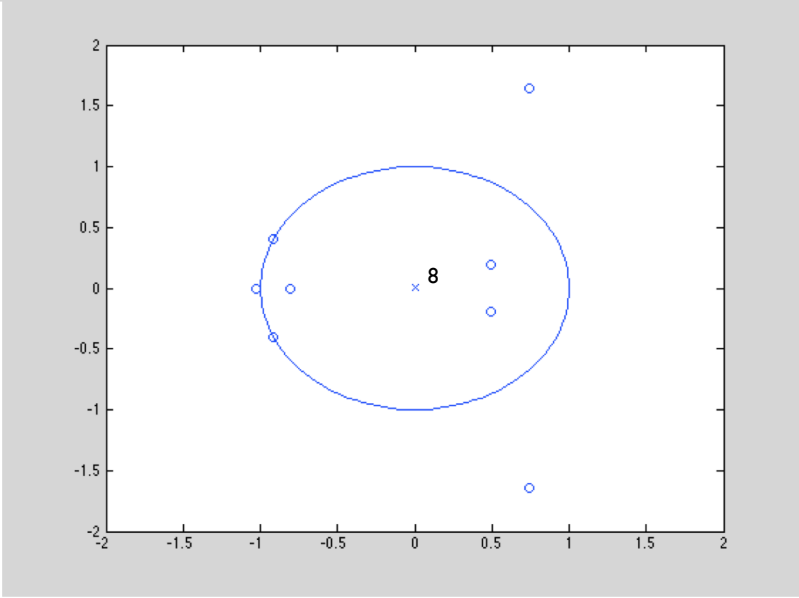
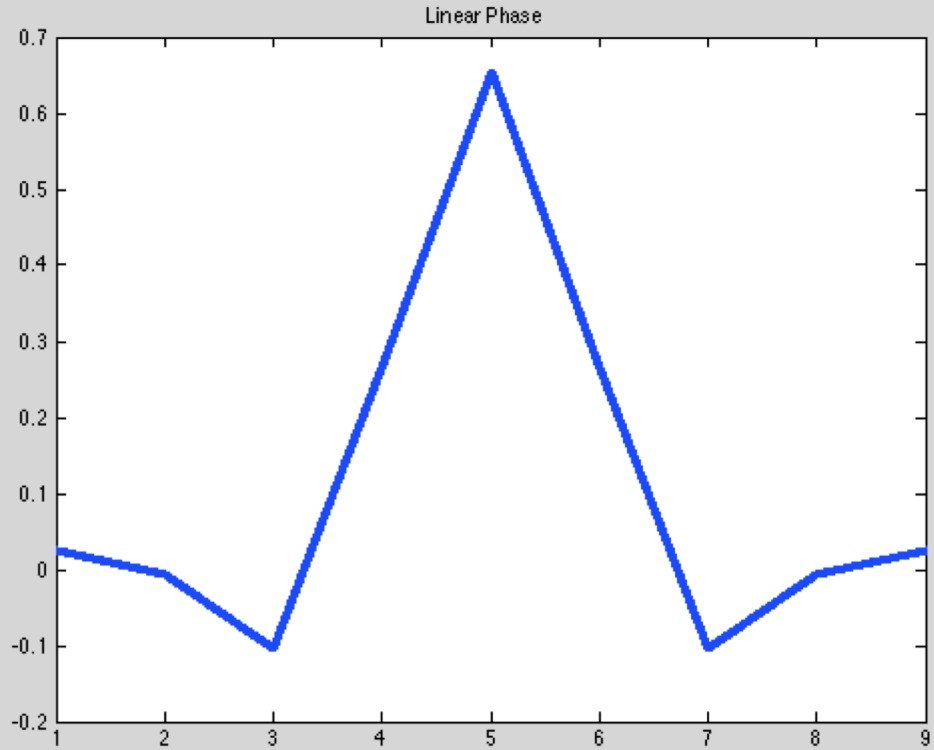
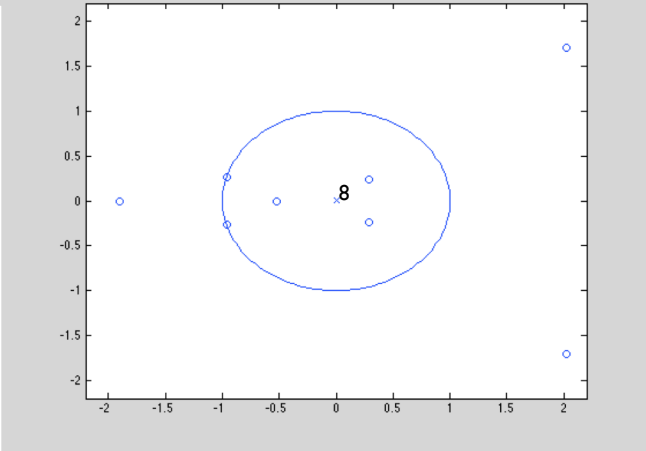

8

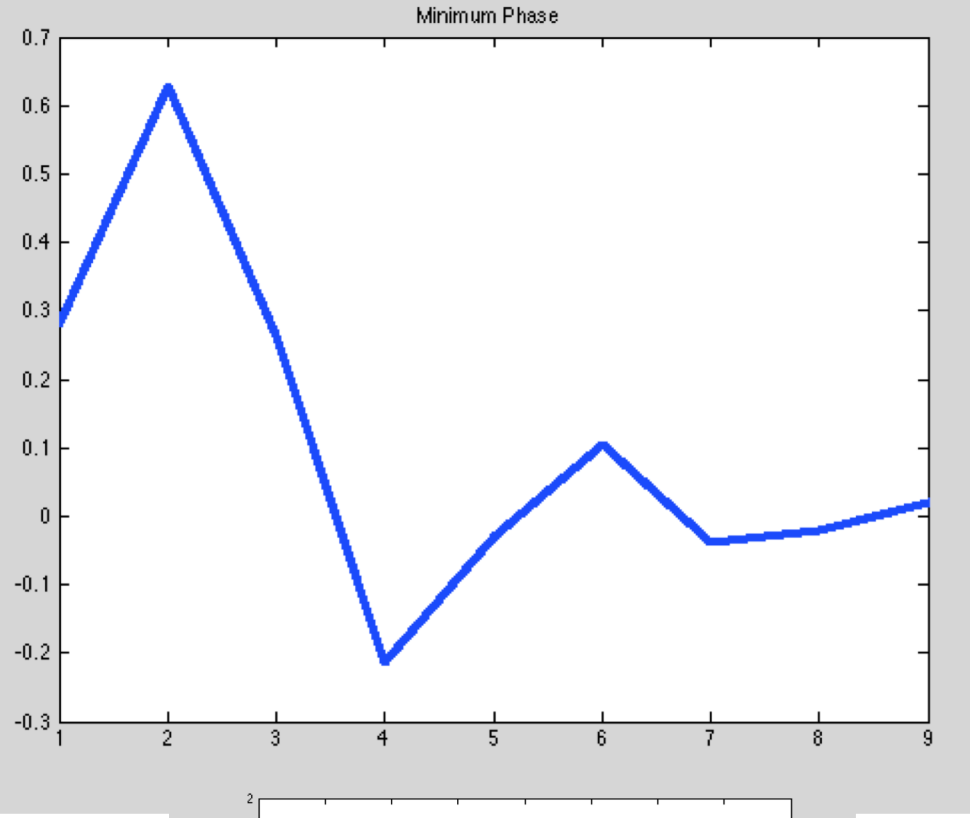
Minimum-phase VS Linear Phase

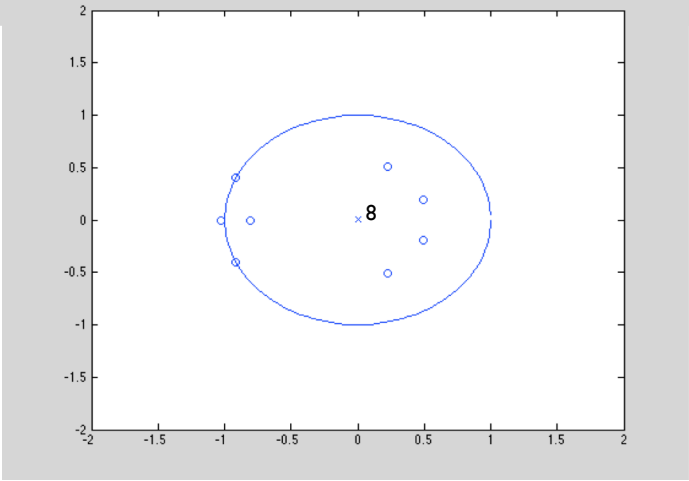
linear phase

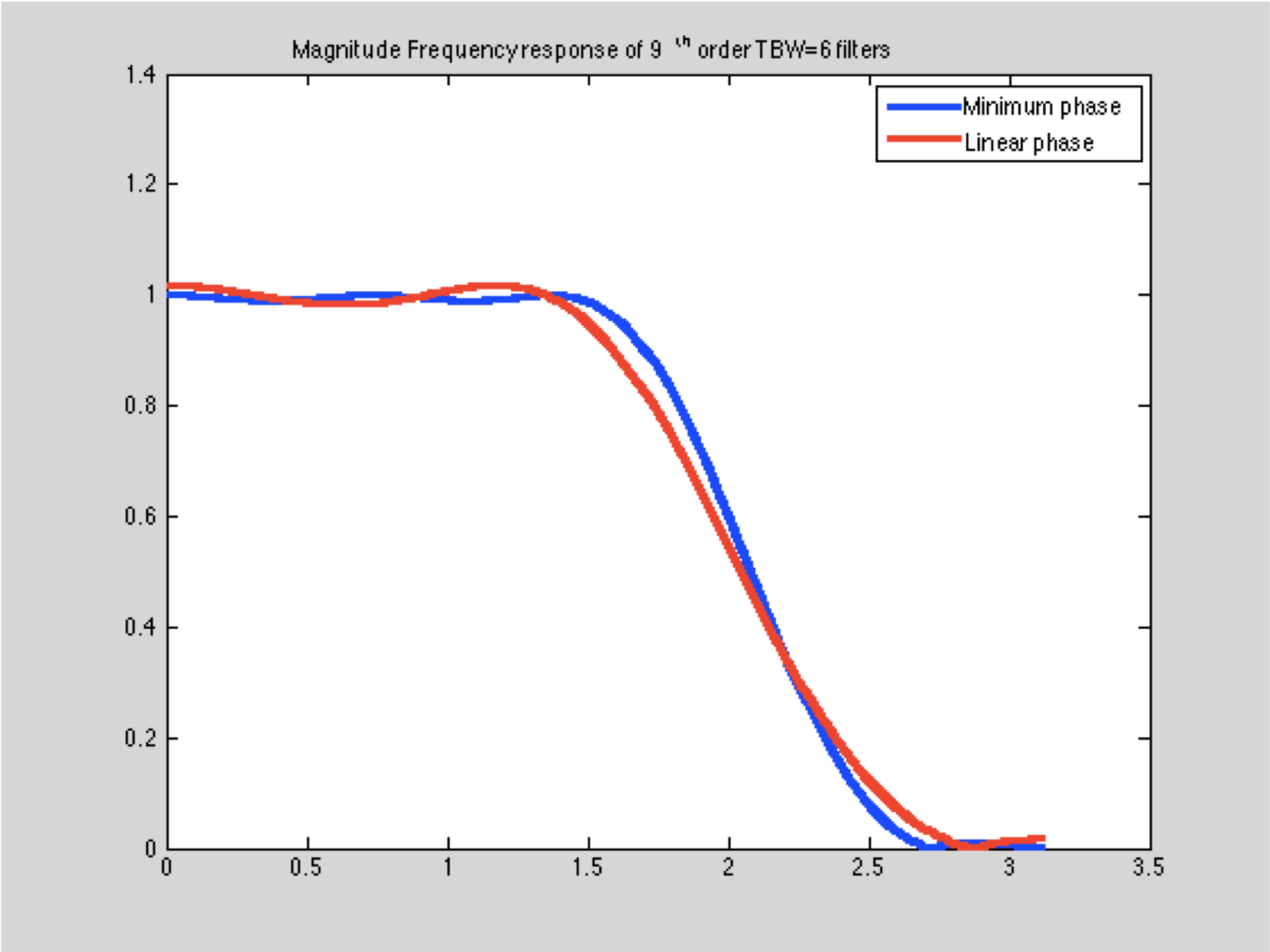




minimum phase






Magnitude Frequency response of 9 th order TBW=6 filters
Minimum phase
Linear phase
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5