EE123
Digital Signal Processing

Lecture 7B
Sampling
What is this Phenomena?

https://www.youtube.com/watch?v=cxddi8m_mzk
Sampling of Continuous Time Signals (Ch. 4)

- **Sampling:**
  - Conversion from C.T (not quantized) into D.T (usually quantized)

- **Reconstruction**
  - D.T (quantized) to C.T

- **Why?**
  - Digital storage (audio, images, videos)
  - Digital communications (fiber optics, cellular...)
  - DSP (compression, correction, restoration)
  - Digital synthesis (speech, graphics)
Sampling of C.T. Signals

- Typical System:

\[ x_c(t) \rightarrow \text{Analog Anti-Aliasing Filter} \rightarrow \text{sampler} \rightarrow x[n] = x_c(nT) \rightarrow \text{Quantizer} \rightarrow \text{ADC A/D} \]

\[ x[n] = x_c(nT) \rightarrow \text{Discrete stuff (DSP, storage....)} \rightarrow y[n] \rightarrow \text{DAC D/A} \rightarrow y_c(t) \rightarrow \text{Reconstruction} \]
Ideal Sampling Model

\[ x_c(t) \xrightarrow{\text{C/D}} x[n] = x_c(nT) \]

define impulsive sampling:

\[ x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots \]
\[ x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT) \]
Ideal Sampling Model

\[ x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

- Not physical: used for modeling & derivations

\[ x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t) \]

- How is \( x[n] \) related to \( x_s(t) \) in freq. domain?
Frequency Domain Analysis

• How is $x[n]$ related to $x_s(t)$ in the Freq. Domain?

$x_s(t) \quad : \text{C.T}$

$x[n] \quad : \text{D.T}$

$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$X_s(j\Omega) = \sum_n x_c(nT)e^{-j\Omega nT}$

$X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}$

$\omega = \Omega T$

$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega = \omega/T}$

$X_s(j\Omega) = X(e^{j\omega})|_{\omega = \Omega T}$
Frequency Domain Analysis

• How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \sum_{n} \delta(t - nT)$$

$\Delta \equiv s(t)$
Frequency Domain Analysis

• How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \sum_n \delta(t - nT)$$

\[= s(t)\]

\[= \frac{1}{T} \sum_n \delta\left(\frac{t}{T} - n\right) = \frac{1}{T} \downarrow \downarrow \downarrow \uparrow \downarrow \uparrow \left(\frac{t}{T}\right)\]

Recall $\downarrow \downarrow \downarrow \uparrow \downarrow \uparrow (t) = \frac{1}{T} \delta(t)$
Frequency Domain Analysis

- How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \sum_{n} \delta(t - nT) \triangleq s(t)$$

$$s(t) \leftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k) \equiv \Omega_s$$
Frequency Domain Analysis

\[
X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)
= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T}
\]

• \(X_s\) is replication of \(X_c\)!
Frequency Domain Analysis

\[ s(t) \rightarrow \cdots \rightarrow \frac{2\pi}{T} \rightarrow \cdots \]

So, if:

\[ X_c(j\Omega) \quad \text{and} \quad \Omega_s > 2\Omega_N \]

\[ X_s(j\Omega) \]

\[ \omega = \Omega T \]

\[ X(e^{j\Omega}) \]

\[ \frac{\Omega_s}{2} \cdot T = \pi \]
So, if:

- \( X_c(j\Omega) \)
- \( \Omega_s < 2\Omega_N \)
- \( \omega = \Omega T \)
- \( \frac{\Omega_s}{2} \cdot T = \pi \)
Aliasing

Q: What is the difference in acquisition between the two images?
SNR can be assessed again using the difference method. Similarly, two disjoint Poisson datasets can be created and reconstructed with SPIRiT using the difference method. The k-space data can then be subsampled by a factor of two in a Cartesian fashion in two ways, yielding two disjoint images. The k-space data will be acquired twice in interleaved fashion for each phase. For each sequence, the central calibration portion of k-space will be repeated.

### Design:
Each patient will undergo an MRI protocol as shown in Table 2. MRI protocol to validate techniques specified in Ref. 1. The protocol will be performed as shown in Figure 36.

### Table 2:
<table>
<thead>
<tr>
<th>Technique</th>
<th>Ref. 1</th>
<th>Ref. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRI protocol</td>
<td>Dimilin</td>
<td>Dimin</td>
</tr>
<tr>
<td>MRI protocol</td>
<td>Full sampled</td>
<td>Full sampled</td>
</tr>
<tr>
<td>MRI protocol</td>
<td>Full sampled</td>
<td>Full sampled</td>
</tr>
<tr>
<td>MRI protocol</td>
<td>1mm, SPIRiT</td>
<td>6.3x, SPIRiT</td>
</tr>
<tr>
<td>MRI protocol</td>
<td>3 spaces, ARC</td>
<td>3 spaces, ARC</td>
</tr>
</tbody>
</table>

Note restored delineation of growth plate and length for routine use now with qin fold acceleration.

Figure 37:
Representative images from nD T1-weighted scan at 1 mm, 6.3x, ARC. Images are too noisy.

Figure 38:
Representative images from nD T2-weighted scan at 1 mm, 6.3x, ARC.
Nyquist Sampling Thm: suppose $x_c(t)$ is bandlimited

$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

if $\Omega_s > 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$

Bandlimitedness is the key to uniqueness

multiple signals go through the samples, but only one is bandlimited!
Reconstruction in Frequency Domain

\[ x[n] \rightarrow \text{Convert to impulse train} \rightarrow x_s(t) \rightarrow H_r(j\Omega) \rightarrow x_r(t) \]

\[
X_s(j\Omega) \]

\[
H_r(j\Omega) \]

\[
X_r(j\Omega) \]

M. Lustig, EECS UC Berkeley
Reconstruction in Time Domain

\[ h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \]

\[ = T \frac{1}{2\pi} \left. e^{j\Omega t} \right|_{-\Omega_s/2}^{\Omega_s/2} \]

\[ = T \frac{e^{j\frac{\Omega_s}{2} t} - e^{-j\frac{\Omega_s}{2} t}}{\pi t} \]

\[ = \frac{T}{\pi t} \sin\left(\frac{\Omega_s}{2} t\right) = \frac{T}{\pi t} \sin\left(\frac{\pi}{T} t\right) \]

\[ = \text{sinc}\left(\frac{t}{T}\right) \]
Reconstruction in Time Domain

\[ x_r(t) = x_s(t) \ast h_r(t) = \left( \sum_n x[n] \delta(t - nT) \right) \ast h_r(t) = \sum_n x[n] h_r(t - nT) \]

The sum of “sincs” gives \( x_r(t) \Rightarrow \) Unique signal

bandlimited by \( \Omega_s \)

M. Lustig, EECS UC Berkeley
• If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

\[
X_r(j\Omega) = \begin{cases} 
TX_s(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\
0 & \text{otherwise}
\end{cases}
\]
Anti-Aliasing

\[ x_c(t) \xrightarrow{\text{Analog}} \text{Anti-Aliasing Filter} H_{LP}(j\Omega) \xrightarrow{\text{sampler}} x[n] = x_c(nT) \xrightarrow{\text{Quantizer}} \]

ADC A/D

\[ t = nT \]

\[ X_c(j\Omega) \]

and \( \Omega_s < 2\Omega_N \)

\[ X_s(j\Omega) \]

\[ X_c(j\Omega) H_{LP}(j\Omega) \]

and \( \Omega_s < 2\Omega_N \)
Non Ideal Anti-Aliasing

\[ X_c(j\Omega) H_{LP}(j\Omega) \]

Graph showing the relationship between \( X_c(j\Omega) H_{LP}(j\Omega) \) and \( \Omega_N \) and \( \Omega_s/2 \). The graph illustrates interference effects across different frequency ranges.

Graph at the bottom shows the function \( X(e^{j\Omega}) \) with frequency ranges from \(-\pi\) to \(\pi\).