Continuous R.V.

Sample space

pdf = prob. dist. fn.

R.V.

pmf = prob. mass function

\[ p(x) \]

\[ \alpha \quad \beta \quad c \]

R.V.

\[ P_x(x_0) \]

\[ x_{\text{min}} \quad x_{\text{max}} \]

Real line
Def: pdf of R.V. \( x \) whose event space is the real line \( x_0 \to -\infty \) to \( x_0 \to +\infty \).

\[
\Pr(a \leq x \leq b) = \Pr(a < x < b) = \Pr(a \leq x \leq b) = \Pr(a \leq x < b) = \int_a^b f_x(x_0) \, dx_0
\]

Event \( |x| < 1 \):

\[
\Pr(|x| < 1) = \int_{-1}^1 f_x(x_0) \, dx_0
\]

\[
\Pr\left(\left[ x', x'+b \right]\right) = \int_{x'}^{x'+b} f_x(x_0) \, dx_0 \approx bf_x(x')
\]
Constraints on pdf:

1. \( \int_{-
fty}^{+
fty} f_{x}(x_{0}) \, dx_{0} = 1 \) \quad \text{normalization property.}

2. \( f_{x}(x_{0}) \geq 0 \) \quad \text{non-negative.}
Wheel of fortune a mushroom between 0 & 1.

\[
\begin{align*}
\int f(x) \, dx &= \begin{cases} 0 & x \leq 0, \\
(i = 1) \frac{1}{2} & x > 0,
\end{cases} \\
\text{otherwise.}
\end{align*}
\]

\[
\begin{align*}
\int f(x) \, dx &= 1, \\
T &= 0 < x < 1, \\
\text{Ex} &
\end{align*}
\]
Expectation

\[ E(x) = \int_{-\infty}^{+\infty} x f_x(x) \, dx \]

Expected value of a fn of a r.v.

\[ g(x) = y \]

\[ E[g(x)] = \int_{-\infty}^{+\infty} g(x_0) f_x(x_0) \, dx_0 \]

Var [x] = \[ E[(x - \overline{x})^2] \]

\[ = \int_{-\infty}^{+\infty} (x_0 - \overline{x})^2 f_x(x_0) \, dx_0 \]

Var [x] = \[ E(x^2) - E(x)^2 \]

Var [x] \geq 0
\[ y = ax + b \implies \]
\[ E(y) = a \ E(x) + b \]
\[ \text{Var}(y) = a^2 \ \text{Var}(x) \]

\[ E(x) = \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} x \cdot f_x(x) \, dx \]
\[ = \int_{a}^{b} x \cdot \frac{1}{b-a} \, dx = \frac{1}{b-a} \left[ \frac{1}{2} x^2 \right]_{a}^{b} = \frac{a+b}{2} \]
\[ E(x^2) = \int_a^b x^2 \, dx = \frac{1}{b-a} \int_a^b (\frac{x}{3})^2 \, dx \]

\[ \Rightarrow 0 < x \leq \frac{b-a}{2} \]

\[ \frac{\int_a^b x^2 \, dx}{b-a} = \frac{1}{2} \left( \frac{b-a}{2} \right)^2 \]

\[ \Rightarrow x > \frac{b-a}{2} \]

\[ E(x^2) = \int_a^b x^2 \, dx = \int_{\frac{b-a}{2}}^{b-a} x^2 \, dx \]

\[ \text{Note: } \int_0^a e^{-x} \, dx = 1 \]

\[ \int_0^b e^{-x} \, dx \to A \]

\[ \text{Otherwise, } x \geq \frac{b-a}{2} \]

\[ \Rightarrow \int_{\frac{b-a}{2}}^{b-a} x^2 \, dx = \int_0^{b-a-x_0} x^2 \, dx + \int_{b-a-x_0}^{b-a} x^2 \, dx \]

\[ \text{Note: } \int_0^a e^{-x} \, dx = 1 \]

\[ \int_0^b e^{-x} \, dx \to A \]
$E(x) = \frac{1}{x}$

$\text{Var}(X) = \frac{1}{x^2}$

$s.d. = \frac{1}{x}$

Exp. R.V., mean = s.d.

$f(x)(x_0) \to \exp. \to e^{-\lambda x}$

$p_r(x > a) = \int_a^{\infty} e^{-\lambda x} \, dx = e^{-\lambda a}$

EX: Time to bulb failure is exp.

mean 10 days.

Right now = midnight. Prob (failure 6:00am, 6:00pm) at same day.
\[ p \left( \frac{1}{4} < x < \frac{3}{4} \right) = p(x > \frac{1}{4}) - p(x > \frac{3}{4}) \\
= e^{-\frac{1}{40}} - e^{-\frac{3}{40}} \\
= 0.0476 \]
CDF = Cumulative Distribution Function

CDF is integral of pdf.

\[ F_X(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f_X(x) \, dx \]

\[ \sum_{k \leq x} P_X(k) \]

\[ \sum \]

\[ X \text{ continuous} \]
\[
\begin{align*}
F_{X}(x) & \\
Pr(X \leq 2) &= \alpha = F_{X}(2) = a \\
Pr(X \leq 4) &= F_{X}(4) = a + b = 1 \\
F_{X}(7) &= Pr(X \leq 7) = \int_{0}^{7} f_{X}(x) \, dx
\end{align*}
\]
PDF

CDF

\[
P_x(x_0) = \frac{d}{dx} F_x(x_0)
\]

CDF

\[
F_x(b) = P_x(x \leq b)
\]

\[
= \int_{-\infty}^{b} f_x(x_0) \, dx
\]

= \int_{a}^{b} f_x(x_0) \, dx

a \quad b
Prop of CDF

\[ F_X(x) = P_r(X \leq x) \]

1. \( F_X(x) \) is monotonically non-decreasing:
   \[ \frac{dF}{dx} \geq 0 \]

2. \( F_X(x) \): if \( x \leq y \),
   \[ F_X(x) \leq F_X(y) \]
Life time of a component \( P(t) \)

\[
\int_{t}^{5/16} f(t) \, dt
\]

1. What is the probability that the component fails in the second month?

\[
Pr(1 \leq x \leq 2) = \int_{1}^{2} f(x) \, dx = \frac{5}{16}
\]

2. Given it did not fail in the first month, what is the probability it fails in the second month?
Event $A = \text{component failed in 1st month}$

$f_x(x_0|A')$

$\int_1^2 f_x(x_0|A') \, dx_0 = \frac{5}{9}$

Event $B = \text{component fail in 2nd month}$.
\( \text{Event } A = \text{ Component failed in first month} \)

\( \text{Event } B = \text{ 2nd month} \)

\[
P(B | A') = \frac{P(A' | B)}{P(A')} = \frac{P(B)}{P(A')} = \frac{5/16}{9/16} = \frac{5}{9}
\]

\[P(A' | B) = P(B)\]

Event \( B \) is included in Event \( A' \)

\[P(A') = \int_{1}^{4} f_{X}(x_0) \, dx_0 = 9/16\]