Bernoulli Trial

\[ P_x(x_0) = \begin{cases} 1 - P & \text{failure}, \\ P & x_0 = 1 \text{ success}, \\ 0 & \text{otherwise}. \end{cases} \]

\[ M_x(s) = 1 - P + e^s P = \sum_{x_0} p_x(x_0) e^{s x_0} \]

\[ E(x) = P \]

\[ \sigma_x^2 = P(1 - P) \]

Bernoulli Process

Def: Series of independent Bernoulli trials each with the same prob of success.
- n indp. Bern. trials.
  \[ k = \text{R.V. } \# \text{ of successes in } n \text{ trials.} \]
  \[ k = \text{sum of } n \text{ indp. Bern. R.V.} \]

What is \( P_k(k_0) \)?

*Method 1:*

\[
M_k(s) = \left[ M_x(s) \right]^n = (1 - P + e^s P)^n
\]

Recall: \((a + b)^n = \sum_{l=0}^{n} \binom{n}{l} a^{n-l} b^l\)

\[
M_k(s) = \sum_{l=0}^{n} \binom{n}{l} P^l (1 - P)^{n-l}
\]

\[
M_k(s) = P_k(0) + e^s P_k(1) + P_k(2) e^{2s} + \cdots
\]

\[
\text{Equate coeff of power of } e^s
\]

\[
P_k(k_0) = \binom{n}{k_0} P^{k_0} (1 - P)^{n-k_0}
\]
\[
\binom{n}{k_0} = \frac{n!}{(n-k_0)! \cdot k_0!}
\]

Method 2: Look into sequential sample space.

In an experiment consisting of \( n \) independent trials,

\[
\{S_n, F_n\} = \begin{cases} 
\text{success} \text{ on } n\text{th trial}.
\end{cases}
\]

\[
\begin{align*}
\left( \begin{array}{c}
P \\
S_1 \\
F_1 \\
1-p \\
1-p \end{array} \right) & \left( \begin{array}{c}
p \\
S_2 < \\
1-p \\
F_2 < \\
1-p \end{array} \right)
\end{align*}
\]
outcome of exactly $k_0$ successes at a leaf of
the tree $= p^{k_0} (1-p)^{n-k_0}$

- But, there are $\binom{n}{k_0}$ such leaves.

$$P_k(k_0) = \binom{n}{k_0} p^{k_0} (1-p)^{n-k_0} \quad k_0 = 0, 1, 2, \ldots$$

Binomial R.V.

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how to compute mean + variance.

3 methods  

1. directly use def. transform

2. $K = \text{sum of} \ n \ R.V. \ X_i \ i = 1, \ldots, n$  

3. $X_1, X_2, X_3, \ldots, X_n$  

Then $\sum_{i=1}^{n} X_i = K$
\[ E(X) = n \quad E(X) = nP \]
\[ \text{Var}(X) = n \quad \text{Var}(X) = nP(1-P) \]

Ex plot Binomial: \( P = \frac{1}{3} \quad n = 4 \)

\[ P_X(k_0) = \binom{4}{k_0} \left( \frac{1}{3} \right)^{k_0} \left( \frac{2}{3} \right)^{4-k_0} \]
Inter-arrival Times for Bernoulli Process

Def: Success in Bernoulli process is defined as an Arrival

\( l_1 = \) discrete r.v. \# of Bernoulli trials up to and including the first success.

\( l_1 \) \( \sim \) first order inter-arrival time.

taken on discrete values 1, 2, 3, ... , what is pmf for \( l_1 \)?
\[ P_{l_1}(l) = (1-P)^{l-1} P \quad l = 1, 2, \ldots \]

\[ \text{geometric PMF.} \]

\[ \text{Transform: } M_{l_1}(s) = \sum_{l=0}^{\infty} P_{l_1}(l) e^{sl} \]

\[ = \sum_{l=1}^{\infty} P(1-P)^{l-1} e^{sl} \]

\[ = \frac{e^s}{1 - e^s (1-P)} \]
\[ E(e_i) = \left[ \frac{d}{ds} M_{e_i}(s) \right]_{s=0} = \frac{1}{p} \]

\[ \sigma^2 = \frac{1-p}{p^2} \]

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Fresh start: For any given time \( n \),

seq of r.v. \( X_{n+1}, X_{n+2}, \ldots \) (The future process)

is also a Bernoulli process and is independent from \( X_1, \ldots, X_n \) (past process)

Mamoulian: Given no success in first \( m \) trials,

what is the conditional pmf for first success?
rth order Interarrival Time?

\[ l_r = \# \text{ of trials up to and including rth success.} \]

\[ l_1 \xrightarrow{\text{X1}} l_2 \xrightarrow{\text{X2}} \cdots \xrightarrow{\text{Xr}} \cdots \]

\[ l_r = \# \text{ of trials to get rth success.} \]

\[ l_r = \text{sum of r inclp. exponential value of R.V.} \]

\[ M_{l_r}(s) = \left[ M_{l_1}(s) \right]^r = \left[ \frac{e^{-sP}}{1 - e^{-s(1-P)}} \right]^r \]
Need to compute \( P_{lr}(l) \) = pmf for \( lr \).

\( P_{lr}(l) = \text{prob } r \text{th success is at } l \text{th trial.} \)

Given exactly \( r-1 \) successes in the previous \( l-1 \) trials,

Conditional prob of having \( r \)th success on the \( l \)th trial.

\( P_{lr}(l) = \left( \text{Prob of having exactly } r-1 \text{ successes in the first } l-1 \text{ trials.} \right) \times \left( \begin{array}{c}
\text{Given exactly } r-1 \text{ successes in the previous } l-1 \text{ trials,} \\
\text{(Conditional prob of having } r \text{th success on the } l \text{th trial.})
\end{array} \right) \)
\[ P_{l_r}(l) = \binom{l-1}{r-1} P^{l-1} (1-P)^{l-r} \]

\[ E(l_r) = r E(l_1) = \frac{r}{P} \]

\[ \sigma^2(l_1) = r \sigma^2_l = \frac{r(1-P)}{P^2} \]
Fired given out samples of dog food

Ex: Leave a sample at door if there is a dog answer.

\[ \text{Pl door answer} = \frac{3}{4} \]

\[ \text{Pl dog in residence} = \frac{2}{3} \]

\[ \text{Pl dog answer} \times \text{pl dog in residence} = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2} \]

Q: Prob his 3rd call if given away his 1st sample?
Bemolli process = Prob success $P = \frac{1}{2}$
prob failure $= 1-P$

success = handing in a can of dog food
Given that he has given away exactly 4 samples on his first eight calls, what is the conditional probability that he will give away his 5th sample on the 11th call?

\[
(1-P)^2 \cdot P = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}
\]
Pr (given away bin 2nd sample on 5th call) =

\[
P_{l_2}(l) = \binom{l-1}{2-1} p^2 (1-p)^{l-2}
\]

\[
P_{l_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2}
\]

\[
p = \frac{1}{8}
\]
Given that he didn't give away his 2nd sample on his 2nd call, what is the conditional prob. that he will leave his 2nd sample on 5th call?

$$\frac{P(l_2 = 5 \mid l_2 > 2)}{P(l_2 > 2)} = \frac{P(l_2 = 5 \text{ and } l_2 > 2)}{P(l_2 > 2)}$$

$$= \frac{P(l_2 = 5)}{1 - P_{l_2}(2)}$$
\[ \frac{(5+1)}{2-1} \frac{p^2}{(1-p)^{5-2}} = \frac{1}{6} \]

Def needs a new supply immediately after giving away the last can. Stars 2 can.

\[ \text{Prob at least 5 calls before he needs new supply} \]

\[ P(l_2 \geq 5) = 1 - P(l_2 \leq 4) \]
\[ \frac{1}{\sum_{i=1}^{n} \frac{2}{\lambda_i^2}} \]