Ex of Total Prob Thin

- not allowed to exceed 2 rolls.

Ex: Roll a fair 4 sided die.
- if I get 1 or 2, roll again.

Otherwise stop.

Q: Compute \( P(\text{sum of all the rolls is at least 4}) \)

\[
P(A_i) = \frac{1}{4}
\]

- If \( A_1 \), (either 3, or 4 in roll 2) \( \Rightarrow P(B|A_1) = \frac{2}{4} \)

- If \( A_2 \), (2nd roll 2, 3, or 4) \( \Rightarrow P(B|A_2) = \frac{3}{4} \)

- If \( A_3 \), \( P(B|A_3) = 0 \)

- If \( A_4 \), \( P(B|A_4) = 1 \)

\[
P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)
\]

\[= \frac{1}{4} \cdot \frac{2}{4} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{9}{16} \]
Bayes Rule

\[ P(A_1, A_2, A_3) \]

\[ A_1 = \text{malignant} \quad A_2 = \text{non malignant} \quad A_3 = \text{other} \]

\[ P(B|A_i) \]

\[ P(A_1, A_2, A_3) = P(A_1)P(B|A_1) = P(B)P(A_1|B) \]

\[ \Rightarrow P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)} \]
$A_1, A_2, \ldots \text{ are disjoint events. Then from a partition of the sample space}$

\[
P(A_i \mid B) = \frac{P(A_i) \ P(B \mid A_i)}{P(B)}
\]

\[
P(A_i \mid B) = \frac{P(A_i) \ P(B \mid A_i)}{P(A_1) \ P(B \mid A_1) + P(A_2) \ P(B \mid A_2) + \ldots + P(A_n) \ P(B \mid A_n)}
\]

$A_i = \text{cause} \quad B = \text{effect}$
\[ A = \text{aircraft present} \]
\[ B = \text{radar may} \sim \text{yes} \]
\[ P(C \mid B \land A) = 0.99 \]
\[ P(C \mid B \land A^c) = 0.1 \]

\[ P(C \land \lnot B) = P(C) \cdot P(\lnot B) \]

\[ P(C) = \frac{P(C \land B \land A) + P(C \land B \land A^c)}{P(A) \cdot P(B \land A) + P(A^c) \cdot P(B \land A^c)} \]

\[ = \frac{0.05 \times 0.99 + 0.05 \times 0.09}{0.05 \times 0.99 + 0.95 \times 0.09} \]

\[ = 0.3426 \]

\[ P(A \mid B) = \frac{P(A \land B)}{P(B)} \]

\[ = \frac{0.3426}{0.25} \]

\[ = 1.3704 \]

\[ \text{Bayes' effect} \]

\[ \text{Case:} \]

\[ \text{A = aircraft present} \]

\[ \text{B = radar may} \sim \text{yes} \]

\[ \text{P(C | B \land A) = 0.99} \]

\[ \text{P(C | B \land A^c) = 0.1} \]
Ex. 4th Chem.

\[
P(A_i) = \frac{1}{2}
\]

B = winning

\[
P(B|A_i) = 0.3
\]

\[
P(A_i) P(B|A_i) = P(A_i|B) = \frac{P(A_i) P(B|A_i)}{P(B)}
\]

\[
P(B) = \frac{1}{2} + 0.04 + 0.5 \times 0.3 = 0.4
\]

\[
P(A_i|B) = 0.4
\]
Independence

\[ P(A \mid B) = P(A) \]

Math

\[ P(A \mid B) = \frac{P(AB)}{P(B)} \quad \Rightarrow \quad P(AB) = P(A)P(B) \]
2 successive rolls of a 4-sided die.
- All outcomes equally likely. \( P = \frac{1}{16} \)

**Q**

\[ A_i = \{ \text{first roll is } i \} \]
\[ B_j = \{ \text{2nd roll is } j \} \]

\( A_i, B_j \) independent.

**Intuitively** ye.

**Formally**:

\[ P(A_i B_j) = \frac{1}{16} \]
\[ P(A_i) = \frac{1}{4} \]
\[ P(B_j) = \frac{1}{4} \]

\[ P(A_i B_j) = P(A_i) P(B_j) = \frac{1}{4} \times \frac{1}{4} \]

**Q**

\[ A = \{ \text{1st roll is 1} \} \]
\[ B = \{ \text{sum of 2 rolls is 8} \} \]

Intuitively not independent.

\[ P(A | B) = 0 \]
\[ P(A) = \frac{1}{4} \]

\( \Rightarrow \) not independent.
Conclusion if event \( A, B \) are disjoint and both have non-zero prob., then they can NOT be independent.

\[ A = \{ \text{st roll is 1} \} \quad B = \{ \text{sum is 5} \} \]

\( A, B \) independent?

\[ P(A) = \frac{1}{4} \quad B: \left( \begin{array}{l} 1, 4 \\ 2, 3 \\ 3, 2 \\ 4, 1 \end{array} \right) \quad P(B) = \frac{4}{16} = \frac{1}{4} \]

\[ P(AB) = \frac{1}{16} \]

\[ P(AB) = P(A) \cdot P(B) \]

\[ \frac{1}{16} = \frac{1}{4} \cdot \frac{1}{4} \]
Set

\[ A = \{ (2,2), (3,1), (2,2) \} \]

\[ B = \{ (2,2), (2,3), (3,2), (4,1) \} \]

\[ P(AB) = \frac{1}{16} \]

\[ P(A) = \frac{3}{16} \]

\[ P(B) = \frac{5}{16} \]

\[ P(AB) \neq P(A)P(B) \]

\[ P(AB) \neq \frac{15}{128} \]
Conditional Indep.

Two events $A, B$ conditionally indep. given event $C$, if

$$P(AB|C) = P(A|C) \cdot P(B|C)$$

**Note** if $P(A)P(B) = P(AB)$ then

$$P(AB|C) = \frac{P(ABC)}{P(C)}$$

$$P(ABC) = P(C)P(B|C)P(A|BC)$$

$$P(AB|C) = \frac{P(C)P(B|C)P(A|BC)}{P(C)}$$

$$= P(B|C)P(A|BC)$$

Given $C$, chances of $A$ happening is not affected by whether or not $B$ happened.
Ex 2 i) indp. fair coin tosses

$H_1 = \{1^{st} \text{ head}\}$
$H_2 = \{2^{nd} \text{ head}\}$
$D = \{2 \text{ tosses have diff results}\}$

$H_1, H_2$ indp.

Now what if we condition upon $D$.

$P(H_1 | D) = \frac{1}{2}$
$P(H_2 | D) = \frac{1}{2}$

$P(H_1, H_2 | D) \neq P(H_1 | D) \cdot P(H_2 | D)$

$\Rightarrow$ conditional upon $D$, not indp.
Ex 2 coins: blue \( \text{red} \).

Choose at random \( \frac{1}{2} \)

Flip 2 indp tosses.

Blue: \( P(h) = 0.99 \)
\( P(h) = 0.01 \)

Red: \( P(t) = 0.01 \)
\( P(t) = 0.99 \).

\( H_i \): \( i \)th toss is head.

Given B, \( H_1 \) and \( H_2 \) are indep.

\[ P(H_1, H_2 | B) = 0.99 \times 0.99. \]

\[ P(H_1 | B) = 0.99 \]

\[ P(H_2 | B) = 0.99 \]
H1 and H2 unconditionally.

\[
P(H_1) = P(B) P(H_1 | B) + P(B^c) P(H_1 | B^c)
\]

\[
= \frac{1}{2} \times 0.99 + \frac{1}{2} \times 0.01 = \frac{1}{2}
\]

\[
P(H_2) = \frac{1}{2}
\]

\[
P(H_1, H_2) = P(B) P(H_1 H_2 | B) + P(B^c) P(H_1 H_2 | B^c)
\]

\[
= \frac{1}{2} \times 0.99 \times 0.99 + \frac{1}{2} \times 0.01 \times 0.01
\]

\[\text{No}\]