Events $A_1, A_2, ..., A_n$ are independent if:

$$P\left(\cap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i)$$

for every subset $S$ of $\{1, 2, 3, ..., n\}$.

Ex: 3 events $A_1, A_2, A_3$.

What are all possible subsets of $\{1, 2, 3\}$?

$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

$$P(A_1 A_2) = P(A_1) P(A_2)$$
$$P(A_1 A_3) = P(A_1) P(A_3)$$
$$P(A_2 A_3) = P(A_2) P(A_3)$$

$\rightarrow$ pairwise independent.
\[ P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3) \]

For us to call \( A_1, A_2, A_3 \) independent.

**Example:** 2 independent fair coin tosses.

\[ H_1 = \{ \text{1st toss is heads} \} \]

\[ H_2 = \{ \text{2nd toss is heads} \} \]

\[ D = \{ \text{2 tosses have different results} \} \]

\[ \implies P(D) = \frac{1}{4} = \frac{1}{2} \]

**Q:** Are \( H_1, H_2 \) and \( D \) independent?

\( H_1 \) and \( H_2 \) are independent.

\[ P(H_1, H_2) = P(H_1)P(H_2) \]

**Are** \( H_1 \) and \( D \) independent?

\[ P(D \mid H_1) = \frac{P(H_1D)}{P(H_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = P(D) \]

\( \implies \) \( D \) and \( H_1 \) are also independent.
How about: \[ P(H_1, H_2, D) = P(H_1) P(H_2) P(D) \]

\[ \Rightarrow \text{not True} \]

\[ \Rightarrow H_1, H_2, \text{ and } D \text{ are not independent.} \]
Independent Trials

Seq of independent, but identical stages in an experiment.

If 2 outcomes \[\rightarrow\] Bernoulli Trials.

Ex: n independent tosses of a coin.
What is prob is a seq of k heads.
for one seq of k heads and n-k tails

\[ p(\cdot) = p^k (1-p)^{n-k} \]

How many such seq are there?

\[ \binom{n}{k} = \frac{n!}{k! (n-k)!} = \binom{n}{n-k} \]

\[ n \text{ choose } n-k \]

Ex n = 3, k = 2

\[ \frac{3!}{2! 1!} = 3 \]

\[ p(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k} \]

\[ = \frac{n!}{k! (n-k)!} p^k (1-p)^{n-k} \]
\[
\sum_{K=0}^{n} P(K \text{ heads}) = 1 \Rightarrow \\
\sum_{K=0}^{n} \frac{n!}{K! (n-K)!} p^K (1-p)^{n-K} = 1
\]
$1,000,000,000 = 25,000$

\[
\frac{1,000,000,000}{4,000} = 0.001
\]

Q: What is prob

Ex

$C$ modes
$n$ subscribers
Each customer dials up with prob $p$ (indep. of the others)

Q: What is prob that customers get rejected?
Define $p(k) = p(k \text{ subs are correct})$

$p(\cdot) = \sum_{k=C+1}^{n} p(k)$

$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$p(\cdot) = \sum_{k=C+1}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

Ex

$n = 100 \quad p = 0.7 \quad C = 15$

$\Rightarrow p(\cdot) = 0.0399$
counting

- helps in do prob. calc.

Process \( r \) stages.

1. \( n_1 \) possible results in stage 1
2. For ALL possible results of 1st stage, there are \( n_2 \) possible results at stage 2
3. \( n_i \) possible results at stage \( i \) : total # of possibilities = \( n_1 \cdot n_2 \cdots n_r \)
Ex

3 digit phone #,

\[ 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^7 \]

\[ 1 \, 2 \, 3 \, 4 \, 5 \, 6 \, 7 \]

If no 0 or 2

in 1st digit

\[ \frac{1}{8} \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 8 \times 10^6 \]

Selecting \( k \) objects out of \( n \)

Order matters

Permutation

Order doesn't matter

Combination
Choose 2 out of 4 balls, how many different sequences can I get?

\[ 4 \times 3 = 12 \]
n objects choose k

1. First time 1 out of n left with n-1 objects.

2. Second time 1 out of n-1

3. Third time 1 out of n-2

\[ \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+2) \cdot (n-k+1)}{(n-k) \cdot (n-k-1) \cdot (n-k-2) \cdots 2 \cdot 1} \]

# of permutations
\[ \binom{h}{n-k} = \frac{h!}{(n-k)!} \]

**Ex.** How many distinct words are there?

\[ \frac{26!}{22!} = \frac{26 \times 25 \times 24}{23} \]

\[ = 358,000 \]