EE 126 Fall 2006 Midterm #1
Thursday October 6, 7–8:30pm

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

- You have 90 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has three problems that are in no particular order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one single-sided, handwritten, 8.5×11 formula sheet plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Problem	Score
1 [10 points]	
2 [12 points]	
3 [18 points]	
Total	

Problem 1: (10 points)
Consider the following game: first a coin with $\mathbb{P}(\text{heads}) = q$ is tossed once. If the coin comes up tails, then you roll a 4-sided die; otherwise, you roll a 6-sided die. You win the amount of money (in dollars \$) corresponding to the given die roll. Let X be an indicator random variable for the coin toss ($X = 0$ if toss is tails; $X = 1$ if toss is heads), and let Y be the random variable corresponding to the amount of money that you win.

(a) (3pt) Compute the joint PMF $p_{X,Y}$. (It will be a function of q).

(b) (4pt) Compute the conditional PMF $p_{X\,|\,Y}$, again as a function of q. Supposing that it is known that (on some trial of this game) you made 2\$ or less, determine the probability that the initial coin toss was heads, as a function of q.

(c) (3pt) Assume that you have have to pay 3\$ each time that you play this game. Determine, as a function of q, how much money you will win or lose on average. For what value of q do you break even?

Solutions:

(a) (3 pt) We have

$$p_{X,Y}(x,y) = \begin{cases} q/6 & \text{if } x = 1 \text{ and } y \in \{1,2,3,4,5,6\} \text{ (with prob. } q \text{ roll a 6-sided die)} \\ (1-q)/4 & \text{if } x = 0 \text{ and } y \in \{1,2,3,4\} \text{ (with prob. } 1-q \text{ roll a 4-sided die)} \\ 0 & \text{otherwise.} \end{cases}$$

(b) (4 pt) By marginalizing the joint PMF from (a), we first we compute

$$p_Y(y) = \begin{cases} (1-q)/4 + q/6 & \text{if } y \in \{1,2,3,4\}, \\ q/6 & \text{if } y \in \{5,6\} \\ 0 & \text{otherwise.} \end{cases}$$

We then write

$$\begin{aligned} p_{X|Y}(x|y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \begin{cases} \frac{3(1-q)}{(3-q)} & \text{if } x = 0 \text{ and } y \in \{1,2,3,4\} \\ \frac{2q}{3-q} & \text{if } x = 1 \text{ and } y \in \{1,2,3,4\} \\ 1 & \text{if } x = 1 \text{ and } y = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c) (3 pt) Given the set-up of the game, the expected amount that we win is given $\mathbb{E}[Y] - 3$. We compute

$$\begin{aligned} \mathbb{E}[Y] &= \sum_y y p_Y(y) \\ &= q + 5/2. \end{aligned}$$

Therefore, we break even once $\mathbb{E}[Y] = q + 5/2 \geq 3$, or once $q \geq 1/2$.

Problem 2: (12 points)

Suppose one has a deck of cards that are well-shuffled, meaning that each card is equally likely to be located anywhere in the deck, independently of the position of all the other cards.

(a) (2 pt) In how many ways can the cards be shuffled?

Now suppose someone removes cards from the deck, one by one. (In each of the following three parts, assume that we start with a fresh deck each time.)

(b) (3 pt) In how many ways can we remove 7 cards, such that all of those are spades?

(c) (3 pt) In how many ways can we remove 10 cards, such that 4 are spades and 6 are hearts?

(d) (4 pt) If one removes 20 cards, what is the probability that 8 are spades, but 6 are *NOT* hearts?

Solutions:

(a) (2 pt) For different shuffles of the deck, we are looking at ordered sequences of 52 cards, so the total number is $52!$.

(b) (3 pt) First suppose that we do not care about the order. Then the number of ways to remove 7 spades out of 13 is $\binom{13}{7}$. If we now consider ordered sequences, we have to multiply the result by 7!, which gives $\frac{13!}{6!}$ ways.

(c) (3 pt) Again, first let's disregard the order. We want to remove 4 spades out of 13 (which can be done in $\binom{13}{4}$ ways) and 6 hearts out of 13 hearts (which can be done in $\binom{13}{6}$ ways). Now there are 10! ways to rearrange the 10 cards, so in total $10!\binom{13}{4}\binom{13}{6}$ ways.

(d) (4 pt) There are several ways to interpret the question. Here we present the solution for "we remove 20 cards one by one, what is the probability that there are exactly 8 spades, and 6 (or more) are not hearts"

Intially, suppose the cards are not ordered. We want to remove 8 spades out of 13 (which can be done in $\binom{13}{8}$ ways), then 6 clubs or diamonds out of 26 (we remove all the spades and hearts)(in $\binom{26}{6}$ ways), and then the remaining 6 can be anything but spades, so chosen out of $39 - 6$ (we already removed 6 non-spades) (in $\binom{33}{6}$ ways). Total number of ways to remove those, including ordering: $20!\binom{13}{8}\binom{26}{6}\binom{33}{6}$. Since the total number of ways to remove 20 cards is $52!/32!$, the required probability is $\frac{20!32!}{52!}\binom{13}{8}\binom{26}{6}\binom{33}{6}$.

Problem 3: (18 points)

John can either walk to school (which takes 25 min), or take the bus (the bus takes 10 min). However, the buses don't have a fixed schedule. Instead, there is probability p that a bus will arrive on each even-numbered minute (e.g., $t = 0, 2, 4, \ldots$). If John goes to the bus stop, then he always arrives at some odd-numbered minute (e.g., $t = 1, 3, 5 \ldots$). Buses never arrive at an odd-numbered minute.

(a) (2 pt) Let X be a random variable associated with the time between two consecutive buses. Find the expected value $\mathbb{E}[X]$.

(b) (3 pt) What is the expected time it takes to get to school if John goes by bus (including both the waiting time at the bus stop, and driving time)?

Now suppose that John has no idea what p is, so that his strategy is to flip a fair coin: if the coin is heads, he walks, if the coin is tails, he waits for the bus.

(c) (3 pt) Letting Y be the total time it takes to get to school, find the PMF of Y and compute $\mathbb{E}[Y]$.

(d) (3 pt) We are interested in the variance of Y. John's friend Bob gives the following argument: "Let v_1 be the variance of the time needed to go to school if John walks, and v_2 the variance of the time needed if he waits for the bus. Because John has equal chances of walking or taking the bus, the variance of Y is just the average of v_1 and v_2". Is Bob right? Explain why or why not. (In doing so, you are not required to find the variance of X).

For the following two parts, suppose that John always decides to take the bus.

(e) (4 pt) Let Z_{next} be a discrete random variable corresponding to the time (in minutes) that elapses from John's arrival at the bus stop until the next bus comes, and Z_{last} a random variable associated with the time by which John missed the last bus. Compute the expected values $\mathbb{E}[Z_{next}]$ and $\mathbb{E}[Z_{last}]$.

(f) (3 pt) One might that expect $\mathbb{E}[X] = \mathbb{E}[Z_{next}] + \mathbb{E}[Z_{last}]$ (see part (a) for the definition of X). Explain why this is *not* true.

Solutions:

(a) (2 pt) Recall that X is the RV corresponding to the time interval between consecutive buses. The probability that there are k 2-minutes intervals between two consecutive buses is $(1-p)^{k-1}p$, which corresponds to the PMF of a geometric random variable with parameter p. Consequently, the expected number of 2-minute intervals is $1/p$, and the expected value of X is given by $2/p$.

(b) (3 pt) From part (a), the number of 2-minute intervals for which John has to wait has a geometric distribution with parameter p. From John's perspective, the expected time to the next bus is

$2/p - 1$. (We subtract 1 because he arrives at an odd minute, in the middle of the interval.) Overall, his expected travel time to get to school by bus is given by

$$10 + 2/p - 1 \;=\; 9 + 2/p.$$

(c) (3 pt) We compute the PDF of Y by conditioning as follows:

$$\begin{aligned} p_Y(y) &= \mathbb{P}[Y = y \mid \text{walk}]\,\mathbb{P}[\text{walk}] + \mathbb{P}[Y = y \mid \text{bus}]\,\mathbb{P}[\text{bus}] \\ &= \frac{1}{2}\left\{\mathbb{P}[Y = y \mid \text{walk}] + \mathbb{P}[Y = y \mid \text{bus}]\right\}. \end{aligned}$$

Now if John walks, then he is guranteed to take 25 minutes, so that

$$\mathbb{P}[Y = y \mid \text{walk}] \;=\; \begin{cases} 1 & \text{if } y = 25 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if he takes the bus and takes a total time of y, then the number of whole 2-minute intervals that John had to wait for is $1/2(y - 11)$. (Note that y must be odd since he arrives in the middle of an interval.) Consequently, we have

$$\mathbb{P}[Y = y \mid \text{bus}] \;=\; \frac{1}{2}(1-p)^{(y-11)/2}p$$

for $y = 11, 13, 15, \ldots$. Putting together the pieces, we obtain

$$p_Y(y) \;=\; \begin{cases} \frac{1}{2} + \frac{1}{2}(1-p)^7 p & \text{if } y = 25 \\ \frac{1}{2}(1-p)^{(y-11)/2}p & \text{if } y \neq 25,\ y \geq 11 \text{ and odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we compute the expected value again by conditioning

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[Y \mid \text{walk}]\mathbb{P}[\text{walk}] + \mathbb{E}[Y \mid \text{bus}]\,\mathbb{P}[\text{bus}] \\ &= 25/2 + 9/2 + 1/p \\ &= 17 + 1/p. \end{aligned}$$

(d) (3 pt) The argument is invalid because we are dealing with a mixture (*and not a sum*) of two random variables. As an explicit counterexample, consider the case where with probability $1/2$ John walks (say takes 25 minute) and with probability $1/2$ he bikes (which takes 15 minutes). The variance for each choice is 0, yet the total variance of his travel time is clearly non-zero.

(e) (4 pt) Because John arrives in the middle of a two minute interval, and the bus distribution is invariant under reversing time, we have that $E[Z_{next}] = E[Z_{last}] = 2/p - 1$, using our previous results from part (b).

(f) (3 pt) Unless $p = 1$ (meaning that a bus arrives at every even time), the two quantities are different. Note that X is the random variable corresponding to the length of time between two consecutive arrivals, where each interval is chosen randomly (with equal probability) regardless of its length. In contrast, given that the process defining Z_{next} and Z_{last} depends on John's arrival times at the station, in this case the interval between the last and the next bus is *not* chosen uniformly at random. Instead, John is more likely to arrive within a longer interval, so that one would expect that $E[Z_{next}] + E[Z_{last}] > E[X]$.