UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probability and Random Processes

Chapter 7 Problems and Final Review
Fall 2006

Issued: Thursday, December 7, 2006 **Due:** NOT TO BE HANDED IN

Reading: For this problem set: §7.1–7.4

Chapter 7 Problems

Problem 12.1

Imagine that on a particular roulette wheel, $\mathbb{P}(\text{WIN}) = \frac{18}{37}$. Suppose that you play 100 games, and are interested in finding the probability that you win at least half of them.

(a) Write an exact summation for this probability, but do not try to evaluate it explicitly.

(b) Use the central limit theorem to derive a suitable approximation to this probability, and compute the value of your approximation.

Problem 12.2

The test scores of 900 students had the following sample statistics:

Mean: 83 Variance: 36

Use Chebyshev's inequality to bound the probability that a randomly selected student received a test score between 71 and 95 inclusive. Is it likely that at least 600 students scored between 71 and 95 inclusive? Why or why not?

Problem 12.3

Discrete random variable X is equal to 0 with probability 0.5 and otherwise takes on values -2 and 2 with equal probabilities. Let X_1, X_2, ... , be independent identically distributed random variables with the same distribution as X. For each of the following sequences, determine whether each converges in probability to some real number $a \in \mathbb{R}$ and, if so, the limit a to which it converges. Justify your answer.

(a) $Y_n = \max(X_1, X_2, ..., X_n)$.

(b) $Z_n = Y_n - \min(X_1, X_2, \ldots, X_n)$.

(c) $T_n = X_1 + X_2 + ... + X_n$

(d) $A_n = (-1)^{(T_n/2)}$.

Problem 12.4

Wombats and dingos arrive in a Poisson manner to a particular water hole in the Australian Outback. The arrival rates of wombats and dingos are 2 and 4 per hour, respectively. Each animal will stay and drink until the next animal arrives to take over. No other animals visit the water hole.

(a) What is the expected number of animals (wombats or dingos) that visit the water hole in a 24-hour period?

(b) Given that a wombat is currently drinking, what is the probability that the next animal to visit is a dingo?

Crocodile Dundee arrives at the water hole at a random time and leaves the water hole immediately after the $900th$ animal he sees departing the water hole.

(c) How long, on average, will Crocodile Dundee have to wait to see a dingo?

(d) Consider the first dingo that Crocodile Dundee sees. How long, on average, does this dingo spend at the water hole?

(e) What does Chebyshev's inequality tell you about the probability that Crocodile Dundee stays at the water hole for between 140 and 160 hours?

(f) Using the approximation provided by the Central Limit Theorem, what is the probability that Crocodile Dundee stays at the water hole for between 140 and 160 hours?

Problem 12.5

Let X and Y be random variables, and let a and b be scalars; X takes nonnegative values.

(a). Use the Markov inequality on the random variable e^{sY} to show that

$$\mathbb{P}(Y \geq b) \leq e^{-sb} M_Y(s),$$

for every $s > 0$, where $M_Y(s)$ is the transform of Y. (This is the unoptimized Chernoff bound.)

(b). Let Z be a standard normal random variable (zero mean, unit variance). For every $s > 0$, part (a) provides us with an upper bound on $\mathbb{P}(Z \geq b)$. How should s be chosen in order to provide the tightest bound? What is the form of the resulting bound?

Final Review Problems

Problem 12.6

Suppose that a pair of random variables X and Y has a joint PDF that is uniform over the shaded region shown in the figure below:

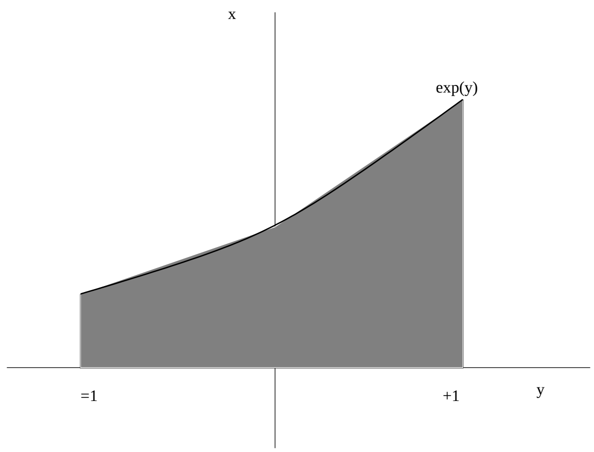


Figure 1: Joint PDF of random variables X and Y.

(a) (7 pt) Compute the Bayes' least squares estimator (BLSE) of X based on Y. (**Note:** You should evaluate the required integrals; however, your answer can be left in terms of quantities like $1/e$ or $\sqrt{2}$).

(b) (7 pt) Compute the linear least squares estimator (LLSE) of X based on Y, as well as the associated error variance of this estimator. Is the LLSE the same as the BLSE in this case? Why or why not? (**Note:** You should evaluate the required integrals; however, your answer can be left in terms of quantities like $1/e$ or $\sqrt{2}$).

(c) (8 pt) Now suppose that in addition to observing some value $Y = y$, we also know that $X \leq 1/e$. Compute the BLSE and LLSE estimators of X based on both pieces of information. Are the estimators the same or different? Explain why in either case.

Problem 12.7

Bob the Gambler: Bob is addicted to gambling, and does so frequently. The time between any two consecutive visits that Bob makes to the local casino can be modeled as an exponentially-distributed random variable X with mean of 1 day. The times between different pairs of consecutive visits are independent random variables. Every time $i = 1, 2, 3 \ldots$ that Bob gambles, he wins/loses a random amount of money modeled as a Gaussian random variable Y_i with mean 0 and variance σ^2. The amounts of money that he wins/loses on different occasions are independent random variables. Suppose that Bob starts off at time $t = 0$ with 0 dollars.

(a) (3 pt) What is the distribution of $N(t)$, the number of times Bob goes to the casino before some fixed time $t \geq 0$?

(b) (2 pt) What is PDF of $M(t)$, the amount of money Bob has won/lost by time t?

(c) (5 pt) Using Chebyshev bound, bound the probability that $M(t)$ is greater that a dollars.

(d) (6 pt) For a given $t > 0$, define a random variable

$$Z_t = \frac{M(t)}{t},$$

corresponding to the average amount of money that Bob has won/lost at time t. As $t \to \infty$, does Z_t converge in probability? If so, prove it. If not, explain intuitively why not.

Problem 12.8

True or false: For each of the following statements, either give a counterexample to show that it is false, or provide an argument to justify that it is true. (**Note:** You will receive no points for just guessing the correct answer; full points will be awarded only when an answer is justified with an example or argument.)

(a) (4 pt) If the Bayes' least squares estimator of X given Y is equal to $\mathbb{E}[X]$, then X and Y are independent.

(b) (4 pt) If random variables X and Y are independent, then they are conditionally independent given any other random variable Z.

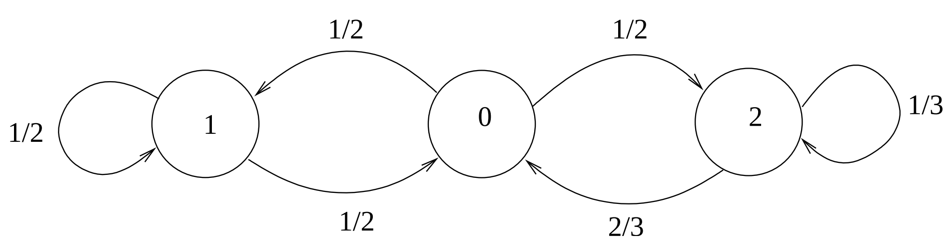
(c) (4 pt) Given a sequence of random variables $\{X_n\}$ such that $\mathbb{E}[X_n] = n$ and $\mathbb{E}[X_n^2] = (n+1)^2$, the sequence $Y_n := \frac{X_n}{n}$ converges in probability to some real number.

(d) (4 pt) If the linear-least squares estimator $\widehat{X}_{LLSE}$ and Bayes' least-squares estimator $\widehat{X}_{BLSE}$ (of X based on Y) are equal, then the random variables X and Y must be jointly Gaussian.

(e) (4 pt) There do not exist any pairs of random variables X and Y with $\mathbb{E}[X^4] = 4$, $\mathbb{E}[Y^4] = 1$ and $\mathbb{E}[X^2 Y^2] = 3$.

Problem 12.9

(20 points) The position of a moving particle can be modeled by a Markov chain on the states $\{0, 1, 2\}$ with the state transition diagram shown below



(a) (3 pt) Classify the states in this Markov chain. Is the chain periodic or aperiodic?

(b) (3 pt) In the long run, what fraction of time does the particle spend in state 1?

(c) (4 pt) Suppose that the starting position X_0 is chosen according to the the steady state distribution. Conditioned on $X_2 = 2$, what is the probability that $X_0 = 0$?

(d) (5 pt) Suppose now we have two particles, both independently moving according to the Markov chain. One particle starts in state 2, and the other particle starts in state 1. What is the average time before at least one of the particles is in state 0?