UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 11
Fall 2006

Issued: Thursday, November 30, 2006 **Due:** Friday, December 8, 2006

Reading: For this problem set: §6.1, 6.2, 6.3

Problem 11.1

(a). Identify the transient, recurrent, and periodic states of the discrete state discrete-transition Markov process described by

$$[p_{ij}] = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

(b). How many classes are formed by the recurrent states of this process?

(c). Evaluate $\lim_{n\to\infty} p_{41}(n)$ and $\lim_{n\to\infty} p_{66}(n)$.

Problem 11.2

Out of the d doors of my house, suppose that in the beginning $k > 0$ are unlocked and $d - k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the d doors. At the end of the day, I leave the door I used that day locked.

(a). Show that the number of unlocked doors at the end of day n, L_n, evolves as the state in a Markov process for $n \geq 1$. Write down the transition probabilities p_{ij}.

(b). List transient and recurrent states.

(c). Is there an absorbing state? How does $r_{ij}(n)$ behave as $n \to \infty$?

(d). Now, suppose that each day, if the door I pick in the morning is locked, I will leave it unlocked at the end of the day, and if it is initially unlocked, I will leave it locked. Repeat parts (a)-(c) for this strategy.

(e). My third strategy is to alternate between leaving the door I use locked one day and unlocked the next day (regardless of the initial condition of the door.) In this case, does the number of unlocked doors evolve as a Markov chain, why/why not?

Problem 11.3

John's office is in a building that has 5 doors. Due to personal peculiarity, John refuses to use the same door twice in a row. In fact, he will choose the door to the left or the right of the last door he used, with probability p and $1 - p$ respectively, and independently of what he has done in the past. For example, if he just chose door 5, there is a probability p that he'll choose door 4, and a probability $1 - p$ that he'll choose door 1 next.

(a) Argue that the above process is a Markov chain.

(b) Find the transition probability matrix.

(c) Find the steady state probabilities.

Problem 11.4

Two thimbles (like tiny cups) are under a dripping roof. At the end of each second, thimble A receives 1 drop of water with probability 1, and thimble B receives 1 drop with probability 2/3 and 0 drops otherwise.

By a complicated automatic mechanism, right before a 4th drop lands in thimble A, both thimbles are emptied. While the thimbles are being emptied, they miss catching the drops that would have otherwise landed inside the thimbles.

(a) Set up a Markov chain model: identify the states, draw the state transition diagram, and indicate the transition probabilities.

(b) If both thimbles were empty when you started watching, what is the probability that both thimbles contain exactly 1 drop after exactly 10,001 seconds?

Problem 11.5

For each of the following definitions of state X_k at time k ($k = 1, 2, \ldots$), determine whether the Markov property is satisfied and, when it is, specify the transition probabilities p_{ij}:

(a) A six-sided die is rolled repeatedly.

- (i) Let X_k denote the largest number rolled in the first k rolls.
- (ii) Let X_k denote the number of sixes in the first k rolls.
- (iii) At time k, let X_k be the number of rolls since the most recent six.

(b) Let Y_k be the state of some discrete-time Markov process at time k (i.e., it is known Y_k satisfies the Markov property) with known transition probabilities q_{ij}.

- (i) For a fixed integer $r > 0$, let $X_k = Y_{r+k}$.
- (ii) Let $X_k = Y_{2k}$.
- (iii) Let $X_k = (Y_k, Y_{k+1})$; that is, the state X_k is defined by the sequence of state *pairs* in a given Markov process.