Problem 5.1
Consider an exponentially distributed random variable $X$ with parameter $\lambda$, and let $F_X$ be the distribution function. Find the real number $\mu$ that satisfies: $F_X(\mu) = \frac{1}{2}$. (Such a number $\mu$ is called the median of the random variable.)

Problem 5.2
Random variables $X$ and $Y$ are uniformly distributed in the region $0 \leq y \leq x \leq 2$. Define the new random variable $T = X + Y$.

(a) Compute the expected values $E(X)$, $E(Y)$, and $E(T)$.

(b) Is $\text{var}(T)$ less than, equal to, or greater than $\text{var}(X) + \text{var}(Y)$? Calculations are not required, but you must justify your answer with a clear argument.

Problem 5.3
(a) Let $Y$ be any continuous random variable for which $\mathbb{E}[|Y|] < +\infty$. Show that

$$
\mathbb{E}[Y] = \int_{0}^{\infty} \mathbb{P}[Y > y] dy - \int_{0}^{\infty} \mathbb{P}[Y < -y] dy.
$$

(b) Let $X$ be any continuous, non-negative random variable for which the expectations $X^n$ exists for integers $n = 1, 2, 3, \ldots$. Show that

$$
\mathbb{E}[X^n] = \int_{0}^{\infty} nx^{n-1} \mathbb{P}[X^n > x] dx.
$$

Hint: You may find the result from part (a) useful. Also it can help to consider a suitably-chosen change of variables.

Problem 5.4
Simulating a continuous random variable: Computers have subroutines that can generate samples of a random variable $X$ that is uniformly distributed in the interval $[0, 1]$. In this problem, we consider how such a subroutine can be used to generate samples of an arbitrary continuous random variable with strictly increasing CDF $G(y)$. Each time $X$ takes a value $x \in (0, 1)$, we generate the unique value $y$ for which $G(y) = x$. (We can neglect the zero probability event that $X$ takes the value 0 or 1.)
(a) Show that the CDF $F_Y(y)$ of the random variable $Y$ thus generated is indeed equal to the given $G(y)$.

(b) Describe how the procedure can be used to simulate an exponential random variable with parameter $\lambda$.

(c) How can the procedure be generalized to simulate a discrete integer-valued random variable?

**Problem 5.5**
A signal $s = 3$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$. When the weather is good, which happens with probability $2/3$, $W$ is normal with zero mean and variance 4. When the weather is bad, $W$ is normal with zero mean and variance 9. In the absence of any weather information:

(a) What is the PDF of $X$?

(b) Calculate the probability that $X$ is between 2 and 4.

**Problem 5.6**
Beginning at time $t = 0$ we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a bulb from company A and a bulb from company B. The lifetime, $X$, of any particular bulb of a particular type is an independent random variable with the following PDF:

For A Bulbs: $f_{X|A}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

For B Bulbs: $f_{X|B}(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(a) Find $P(D)$, the probability that there are no bulb failures during the first $T$ hours of this process.

(b) Given that there are no failures during the first $T$ hours of this process, determine the conditional probability that the first bulb used is a bulb from company A.

(c) Find the expected value and variance for $X$, the time until the first bulb failure.