UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 5
Fall 2006

Issued: Thursday, October 5, 2006 **Due:** Friday, October 13, 2006

Reading: For this problem set: §3.1, 3.2, 3.3, 3.4

Problem 5.1

Consider an exponentially distributed random variable X with parameter λ, and let F_X be the distribution function. Find the real number μ that satisfies: $F_X(\mu) = \frac{1}{2}$. (Such a number μ is called the *median* of the random variable.)

Problem 5.2

Random variables X and Y are uniformly distributed in the region $0 \leq y \leq x \leq 2$. Define the new random variable $T = X + Y$.

(a) Compute the expected values $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{E}(T)$.

(b) Is $\text{var}(T)$ less than, equal to, or greater than $\text{var}(X) + \text{var}(Y)$? Calculations are not required, but you must justify your answer with a clear argument.

Problem 5.3

(a) Let Y be any continuous random variable for which $\mathbb{E}[|Y|] < +\infty$. Show that

$$\mathbb{E}[Y] \;=\; \int_0^\infty \mathbb{P}[Y > y]dy - \int_0^\infty \mathbb{P}[Y < -y]\, dy\,.$$

(b) Let X be any continuous, *non-negative* random variable for which the expectations X^n exists for integers $n = 1, 2, 3, \ldots$. Show that

$$\mathbb{E}[X^n] \;=\; \int_0^\infty n x^{n-1} \mathbb{P}[X^n > x]\, dx.$$

Hint: You may find the result from part (a) useful. Also it can help to consider a suitably-chosen change of variables.

Problem 5.4

Simulating a continuous random variable: Computers have subroutines that can generate samples of a random variable X that is uniformly distributed in the interval $[0, 1]$. In this problem, we consider how such a subroutine can be used to generate samples of an arbitrary continuous random variable with strictly increasing CDF $G(y)$. Each time X takes a value $x \in (0, 1)$, we generate the unique value y for which $G(y) = x$. (We can neglect the zero probability event that X takes the value 0 or 1.)

(a) Show that the CDF $F_Y(y)$ of the random variable Y thus generated is indeed equal to the given $G(y)$.

(b) Describe how the procedure can be used to simulate an exponential random variable with parameter λ.

(c) How can the procedure be generalized to simulate a discrete integer-valued random variable?

Problem 5.5

A signal $s = 3$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$. When the weather is good, which happens with probability 2/3, W is normal with zero mean and variance 4. When the weather is bad, W is normal with zero mean and variance 9. In the absence of any weather information:

(a) What is the PDF of X?

(b) Calculate the probability that X is between 2 and 4.

Problem 5.6

Beginning at time $t = 0$ we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a bulb from company A and a bulb from company B. The lifetime, X, of any particular bulb of a particular type is an independent random variable with the following PDF:

$$\text{For A Bulbs:} \quad f_{X|A}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{For B Bulbs:} \quad f_{X|B}(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathbb{P}(D)$, the probability that there are no bulb failures during the first T hours of this process.

(b) Given that there are no failures during the first T hours of this process, determine the conditional probability that the first bulb used is a bulb from company A.

(c) Find the expected value and variance for X, the time until the first bulb failure.