UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probability and Random Processes

Discussion Notes: Week 12
Fall 2007

Reading: Berstsekas & Tsitsiklis, §5.2, §6.1, §6.2

Key Stuff to Remember:

- **Poisson with parameter $\lambda\tau$:** to describe the number N_τ of arrivals in a Poisson process with rate λ, over a time interval of length τ.

$$p_{N_\tau}(k) = P(k\tau) = e^{\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, \ldots$$

$$\mathbb{E}[N_\tau] = \lambda\tau, \quad \text{var}(N_\tau) = \lambda\tau$$

- **The exponential with parameter λ:** to describe inter-arrival time for a Poisson process with rate λ.

$$f_T(t) = \lambda e^{-\lambda t}, \ t \geq 0, \ \mathbb{E}[T] = \frac{1}{\lambda}, \ \text{var}(T) = \frac{1}{\lambda^2}$$

- **Markov chains** consist of a set of states and a transition matrix p where p_{ij} gives the probability of transitioning to state j from state i, namely:

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j | X_n = i)$$

- **The Chapman-Kolmogorov Equation for the n-Step Transition Probabilities** is an overly-confusing statement of a fairly obvious recurrence (think of computing the probabilities of all possible paths):

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}, \ \text{ for } n > 1, \text{ and all } i, j, \ r_{ij}(1) = p_{ij}$$

Problem 12.1

We are given the following statistics about the number of children in a typical family in a small village:
There are 100 families. 10 have no children; 40 have 1; 30 have 2; 10 have 3; 10 have 4.

(a) If you pick a family at random, what is the expected number of children in that family?

(b) If you pick a child at random (each child is equally likely), what is the expected number of children in that child's family (including the picked child)?

(c) Generalize your approach from part (b) to the case where a fraction p_k of the families has k children, and provide a formula.

Problem 12.2

(a) Identify the transient, recurrent, and periodic states of the discrete state discrete-transition Markov process described by

$$[p_{ij}] = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

(b) How many classes are formed by the recurrent states of this process?

(c) Evaluate $\lim_{n\to\infty} p_{41}(n)$ and $\lim_{n\to\infty} p_{66}(n)$.

Problem 12.3

Out of the d doors of my house, suppose that in the beginning $k > 0$ are unlocked and $d - k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the d doors. At the end of the day, I leave the door I used that day locked.

(a) Show that the number of unlocked doors at the end of day n, L_n, evolves as the state in a Markov process for $n \geq 1$. Write down the transition probabilities p_{ij}.

(b) List transient and recurrent states.

(c) Is there an absorbing state? How does $r_{ij}(n)$ behave as $n \to \infty$?

(d) Now, suppose that each day, if the door I pick in the morning is locked, I will leave it unlocked at the end of the day, and if it is initially unlocked, I will leave it locked. Repeat parts (a)-(c) for this strategy.

(e) My third strategy is to alternate between leaving the door I use locked one day and unlocked the next day (regardless of the initial condition of the door.) In this case, does the number of unlocked doors evolve as a Markov chain, why/why not?