Reading: Berstekas & Tsitsiklis, §1.3, §1.4, §1.5 (no Bayes yet)

Key Stuff to Remember:

- **Conditional Probability:** $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ for $A, B \subseteq \Omega$ and $B \neq \emptyset$.

- **Total Probability:** Given disjoint $A_1, \ldots, A_n$ that partition the sample space $(\bigcup_{i=1}^n A_i = \Omega)$, for any event $B$ we have:

$$P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B) = P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n)$$

- **Independence:** Events $A_1, \ldots, A_n$ are independent if

$$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$$

for every subset $S$ of $\{1, 2, \ldots, n\}$.

**Problem 2.1**
Consider a single throw of two fair, six-sided dice.

(a) Calculate $P(\{\text{“the two faces are the same”}\})$.

(b) Calculate $P(\{\text{“the two faces are the same”}\} \mid \{\text{“sum \leq 3”}\})$.

**Problem 2.2**
(Bertsekas 1.21) Two out of three prisoners are to be released. One of the prisoners asks a guard to tell him the identity of a prisoner other than himself that will be released. The guard refuses with the following rationale: at your present state of knowledge, your probability of being released is $2/3$, but after you know my answer, your probability of being released will become $1/2$, since there will be two prisoners (including yourself) whose fate is unknown and exactly one of the two will be released. What is wrong with the guard’s reasoning?

**Problem 2.3**
Jane has 3 children, each equally likely to be either sex, independently. We define the following events:

- $A = \{\text{“all children are the same sex”}\}$
- $B = \{\text{“there is at most one boy”}\}$
- $C = \{\text{“the children include a boy and a girl”}\}$

(a) Show $A$ is independent of $B$ and $B$ is independent of $C$. 
(b) Is \( A \) independent of \( C \)?

(c) What if boys and girls are not equally likely?

(d) Do these hold if Jane has four children?

**Problem 2.4**
You have five coins: two double-headed, one double-tailed, and two normal. You pick one at random and toss it behind your back.

(a) What is the probability that the *lower* face is heads?

(b) You then open your eyes and see that the upper face is heads. Now what is the probability that the lower face is a head?

(c) You shut your eyes again, toss the coin, and open your eyes to see heads again. Now what is the probability that the lower face is a head?

**Problem 2.5**
*(Galton’s Paradox)* You flip three fair coins. At least two are alike, and it is an evens chance that the third is a head or a tail. Therefore \( P(\{\text{all alike}\}) = 1/2 \). Do you agree?