UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: Probablity and Random Processes

Problem Set 2: Solutions
Fall 2007

Issued: Thursday, September 9, 2007 **Due:** Friday, September 14, 2007

Reading: Bertsekas & Tsitsiklis, §1.3, §1.4. and §1.5

Problem 2.1
You keep taking your driving exam repeatedly, until the first time (if ever) that you pass it. What is the sample space for this experiment? Suppose that your test attempts are independent, and that you have probability p of passing. What is the probability of taking more than 10 attempts to pass?

Solution:

(a) Let P be the outcome that you pass driving exam and F be the outcome you fail. The sample space consists of all sequences that start with 0 or more F and end with P.

$$\Omega = \{P, FP, FFP, FFFP, ...\}$$

(b)You must fail at the first 10 attempts. Thus, the probability is

$$(1-p)^{10}(p + p(1-p) + p(1-p)^2 + \cdots) = (1-p)^{10}$$

Problem 2.2
Let A and B be independent events. Prove the following:

(a) A and B^c are independent.

(b) A^c and B^c are independent.

Solution:

(a)

$$\begin{aligned}
& P(A)P(B^c) \\
= \; & P(A)(1 - P(B)) \\
= \; & P(A) - P(A)P(B) \\
= \; & P(A) - P(A \cap B) \\
= \; & P(A \setminus (A \cap B)) \\
= \; & P(A \cap (A \cap B)^c) \\
= \; & P(A \cap (A^c \cup B^c)) \\
= \; & P((A \cap A^c) \cup (A \cap B^c)) \\
= \; & P(A \cap B^c)
\end{aligned}$$

(b) From (a), we know A and B^c are independent. Then complement A. A^c and B^c are still independent.

Problem 2.3

For each one of the following statements, indicate whether it is true or false, and provide a brief explanation.

(a) If $\mathbb{P}(A \mid B) = \mathbb{P}(A)$, then $\mathbb{P}(B \mid A^c) = \mathbb{P}(B)$.

(b) If 5 out 10 independent fair coin tosses resulted in tails, the events "first toss was tails" and "10th toss was tails" are independent.

(c) If 10 out 10 independent fair coin tosses resulted in tails, the events "first toss was tails" and "10th toss was tails" are independent.

(d) If the events $A_1, \ldots, A_n$ form a partition of the sample space, and if B, C are some other events, then

$$\mathbb{P}(B \mid C) = \sum_{i=1}^{n} \mathbb{P}(A_i \mid C)\mathbb{P}(B \mid A_i).$$

Solution:

1. $\boxed{\text{True}}$
   If $P(A \mid B) = P(A)$, then A and B are independent. And if B is independent of A, then B is also independent of A^c. This implies, by the definition of independence:

$$P(B \mid A^c) = P(B)$$

2. $\boxed{\text{False}}$
   Since there are only 5 tails out of ten, knowledge of one coin toss provides knowledge about the other coin tosses, which means the two events are not independent. In other words, the knowledge that the first coin toss was a tails influences the probability that the tenth coin toss is a tails.

3. $\boxed{\text{True}}$
   Here, all tosses are tails, so knowledge of one coin toss provides no additional knowledge about the tenth coin toss. Therefore the two events are independent.

4. $\boxed{\text{False}}$
   On the left hand side of the expression, since A_i's are disjoint,

$$\begin{aligned} P(B \mid C) &= \frac{P(B \cap C)}{P(C)} \\ &= \sum_{i=1}^{n} \frac{P(A_i)P(B \cap C \mid A_i)}{P(C)} \\ &= \sum_{i=1}^{n} \frac{P(A_i \cap B \cap C)}{P(C)} \end{aligned}$$

However, the right hand side of the given expression shows,

$$\begin{aligned}\sum_{i=1}^{n} P(A_i \mid C)P(B \mid A_i) &= \sum_{i=1}^{n} \frac{P(A_i \cap C)}{P(C)} \frac{P(B \cap A_i)}{P(A_i)} \\ &= \sum_{i=1}^{n} \frac{P(A_i \cap B \cap C)}{P(C)P(A_i)}\end{aligned}$$

where the last line is ONLY TRUE if the events $A_i \cap C$ and $B \cap A_i$ are independent of each other.
Note also for the expression to be true, $i = 1$ and A_1 has to be the entire sample space, i.e. $P(A_1) = 1$. Therefore, the given expression only holds if $A_i \cap C$ and $B \cap A_i$ are independent and $i = 1$.

Problem 2.4
Anne, Betty, Chloe and Daisy were all friends in school. Subsequently each of the six subpairs meet up once; at each of the six meetings, the pair quarrels with some fixed probability p and otherwise the pair retains a firm friendship. Quarrels take place independently of each other. In the future, if any one of the four hears a rumour, she tells it to her firm friends only. Supposing that Anne hears a rumour, what is the probability that:

(a) Daisy hears it?

(b) Daisy hears it given that Anne and Betty have quarrelled?

(c) Daisy hears it given that Betty and Chloe have quarrelled?

(d) Daisy hears it given that she has quarrelled with Anne?

Solution:
The network of friendship is best represented as a square with diagonals, with the corners labelled A (Anne), B (Betty), C (Chloe), and D (Daisy). Each link of the network is absent with probability p, independent of the status of any other link. We write XY for the event that the direct link XY is present, and XY^c when that link is absent. We write $X \leftrightarrow Y$ for the event that person X's rumour is (possibly indirectly) heard by person Y. Also note that all communication in this problem is bidirectional; that is, if person X's rumour can be heard by person Y, then person Y's rumour can also be heard by person X.

1. If the link between A and D is present (event AD), Daisy hears the rumour from Anne directly. If the direct link between A and D is absent (event AD^c, i.e. Anne and Dasiy have quarrelled), Daisy might still hear the rumour depending on the status of the links involving Betty or Chloe. There are $2^6 = 64$ possible network states, implying enumeration of the sample space and summing up outcomes that correspond to $A \leftrightarrow D$

is cumbersome. Rather, we repeatedly apply the Total Probability Theorem and exploit independence between links:

$$
\begin{aligned}
P(A \leftrightarrow D) &= \underbrace{P(A \leftrightarrow D|AD)}_{1}\underbrace{P(AD)}_{1-p} + P(A \leftrightarrow D|AD^c)\underbrace{P(AD^c)}_{p} \quad , \\
P(A \leftrightarrow D|AD^c) &= P(A \leftrightarrow D|AD^c \cap BC)\overbrace{P(BC|AD^c)}^{1-p} + \\
&\quad P(A \leftrightarrow D|AD^c \cap BC^c)\underbrace{P(BC^c|AD^c)}_{p} \quad ,
\end{aligned}
$$

Given $AD^c \cap BC$, note that the event $A \leftrightarrow D$ is still true provided that at least one of the links AB and AC is present and at least one of the links BD and CD is present. In terms of the events and set operations, we require $(AB \cup AC) \cap (BD \cup CD)$ for $A \leftrightarrow D$ to still be true given $AD^c \cap BC$. Therefore, again relying on independence,

$$
\begin{aligned}
P(A \leftrightarrow D|AD^c \cap BC) &= P((AB \cup AC) \cap (BD \cup CD)) = P(AB \cup AC)P(BD \cup CD) \\
&= (1 - P(AB^c \cap AC^c))(1 - P(BD^c \cap CD^c)) \\
&= (1 - \underbrace{P(AB^c)P(AC^c)}_{p^2})(1 - \underbrace{P(BD^c)P(CD^c)}_{p^2}) = \left(1 - p^2\right)^2 \quad .
\end{aligned}
$$

Given $AD^c \cap BC^c$, note that the event $A \leftrightarrow D$ is still true provided that both of the links AB and BD are present or both of the links AC and CD are present. In terms of the events and set operations, we require $(AB \cap BD) \cup (AC \cap CD)$ for $A \leftrightarrow D$ to still be true given $AD^c \cap BC^c$. Thus,

$$
\begin{aligned}
P(A \leftrightarrow D|AD^c \cap BC^c) &= P((AB \cap BD) \cup (AC \cap CD)) \\
&= 1 - P((AB \cap BD)^c \cap (AC \cap CD)^c) \\
&= 1 - P((AB \cap BD)^c)P((AC \cap CD)^c) \\
&= 1 - (1 - P(AB \cap BD))(1 - P(AC \cap CD)) \\
&= 1 - (1 - \underbrace{P(AB)P(BD)}_{(1-p)^2})(1 - \underbrace{P(AC)P(CD)}_{(1-p)^2}) \\
&= 1 - \left(1 - (1-p)^2\right)^2 \quad .
\end{aligned}
$$

Finally, substituting these previous two answers into the above equations yields

$$
\boxed{P(A \leftrightarrow D) = 1 - p + \left\{\left(1 - p^2\right)^2 (1-p) + \left[1 - \left(1 - (1-p)^2\right)^2\right] p\right\} p}
$$

2. Using all the similar reasonings discussed in part (a),

$$
\begin{aligned}
P(A \leftrightarrow D|AB^c) &= \overbrace{P(A \leftrightarrow D|AB^c \cap AD)}^{1}\overbrace{P(AD|AB^c)}^{1-p} + \\
&\quad P(A \leftrightarrow D|AB^c \cap AD^c)\underbrace{P(AD^c|AB^c)}_{p}
\end{aligned}
$$

$$
\begin{aligned}
P(A \leftrightarrow D | AB^c \cap AD^c) &= P(AC \cap (CD \cup (BC \cap BD))) \\
&= P(AC)\,(1 - P(CD^c \cap (BC \cap BD)^c)) \\
&= (1-p)\,(1 - P(CD^c)(1 - P(BC \cap BD))) \\
&= (1-p)\left(1 - p\left(1 - (1-p)^2\right)\right)
\end{aligned}
$$

$$
\boxed{P(A \leftrightarrow D | AB^c) = 1 - p + (1-p)\left\{1 - p\left[1 - (1-p)^2\right]\right\} p} \quad .
$$

3. Again reasoning as in part (a), where we already computed $P(A \leftrightarrow D | AD^c \cap BC^c)$,

$$
\begin{aligned}
P(A \leftrightarrow D | BC^c) &= \underbrace{P(A \leftrightarrow D | BC^c \cap AD)}_{1}\underbrace{P(AD)}_{1-p} + \underbrace{P(A \leftrightarrow D | AD^c \cap BC^c)}_{1-(1-(1-p)^2)^2}\underbrace{P(AD^c)}_{p} \\
&= \boxed{1 - p + \left\{1 - \left[1 - (1-p)^2\right]^2\right\} p} \quad .
\end{aligned}
$$

4. All the required calculations were done in part (a):

$$
\boxed{P(A \leftrightarrow D | AD^c) = \left(1 - p^2\right)^2 (1-p) + \left[1 - \left(1 - (1-p)^2\right)^2\right] p} \quad .
$$

Problem 2.5

Imagine that you have n drawers in your filing cabinet, and that you left your term paper in drawer k with probability D_k. Furthermore, suppose that these drawers are so messy, that even if you correctly guess that the term paper is in drawer j, the probability that you find it is p_j. Suppose you search for your paper in a particular drawer, say drawer i, and the search is unsuccessful. Conditional on this event, show that the probability that your paper is in drawer j is:

(a) $\frac{D_j}{1-p_i D_i}$, if $j \neq i$, and

(b) $\frac{D_i\,(1-p_i)}{1-p_i D_i}$ if $j = i$.

Solution:

Define the following events:

A is the event that you search drawer i and find nothing,
B is the event that you search drawer i and find the paper,
C_k is the event that the paper is in drawer k, $k = 1, \ldots, n$.

We have

$$
P(B) = P(B \cap C_i) = P(B \mid C_i)P(C_i) = p_i D_i,
$$

and therefore

$$
P(A) = 1 - P(B) = 1 - p_i D_i.
$$

1. If $j \neq i$, then $P(A \cap C_j) = P(C_j)$, and we have

$$P(C_j \mid A) = \frac{P(A \cap C_j)}{P(A)} = \frac{P(C_j)}{P(A)} = \frac{D_j}{1 - p_i D_i},$$

as required.

2. Now suppose $j = i$. We have

$$P(C_j \mid A) = \frac{P(A \cap C_j)}{P(A)}.$$

Now, recall that

$$P(A \mid C_j) = \frac{P(A \cap C_j)}{P(C_j)},$$

which implies that

$$P(A \cap C_j) = P(C_j) \cdot P(A \mid C_j) = D_i(1 - p_i).$$

Combining the above relations, we have

$$P(C_j \mid A) = \frac{D_i(1 - p_i)}{1 - p_i D_i}.$$

We could calculate this in an alternative manner using part (a) and the fact that

$$P(C_i \mid A) = 1 - \sum_{j \neq i} P(C_j \mid A).$$

We then have

$$\begin{aligned} P(C_i \mid A) &= 1 - \sum_{j \neq i} \frac{D_j}{1 - p_i D_i} \\ &= \frac{1 - p_i D_i - \sum_{j \neq i} D_j}{1 - p_i D_i} \\ &= \frac{D_i - p_i D_i}{1 - p_i D_i} \qquad \text{since } D_i + \sum_{j \neq i} D_j = 1. \end{aligned}$$

Problem 2.6

Three persons roll a fair n-sided die once. Let A_{ij} be the event that person i and person j roll the same face. Show that the events A_{12}, A_{13}, and A_{23} are pairwise independent but are not independent.

Solution:

The probability that persons 1 and 2 both roll a particular face is $1/n^2$. Therefore,

$$P(A_{12}) = P(A_{13}) = P(A_{23}) = \frac{n}{n^2} = \frac{1}{n}.$$

Similarly, we also have

$$P(A_{12} \cap A_{13}) = P(\text{all players roll the same face}) = \frac{n}{n^3} = \frac{1}{n^2},$$

so

$$P(A_{12} \cap A_{13}) = P(A_{12}) \cdot P(A_{13})$$

. Hence A_{12} and A_{13} are independent, and the same is true of any other pair from the events A_{12}, A_{13}, and A_{23}. However, A_{12}, A_{13}, and A_{23} are not independent. In particular, if A_{12} and A_{13} occur, then A_{23} also occurs.

Problem 2.7

Larry and Consuela love to challenge each other to coin flipping contests. Suppose that Larry brings $2n+1$ fair coins to school, and assume that he lets Consuela play with $n+1$ coins, and he plays with the remaining n coins. Show that the probability that after all the coins have been flipped Consuela will have gotten more heads than Larry is $\frac{1}{2}$.

Solution:

Consider the flips of the first n coins by each player. At this point of the experiment, Larry is finished, and Consuela has one more coin to flip.

Let M be the event that Consuela obtains more heads than Larry after these n flips, E be the event that she and Larry have obtained the same number of heads, and L be the event that she obtains fewer heads than Larry. Since the coins are fair, $P(M) = P(L)$, so that $P(E) = 1 - P(M) - P(L) = 1 - 2P(M)$.

Now consider Consuela's last flip, and let C be the event that she obtains more heads than Larry in the completed experiment. Note that

$$\begin{aligned} P(C|M) &= 1 \\ P(C|E) &= 1/2 \\ P(C|L) &= 0 \end{aligned}$$

Since the events M, E, and L are mutually exclusive and collectively exhaustive, we may write

$$\begin{aligned} P(C) &= P(M)P(C|M) + P(E)P(C|E) + P(L)P(C|L) \\ &= P(M) + (1 - 2P(M))(1/2) \\ &= \boxed{1/2} \end{aligned}$$