Problem Set 3
Fall 2007

Issued: Thursday, September 13, 2007
Due: Friday, September 21, 2007

Reading: Bertsekas & Tsitsiklis, §1.4, §1.5. and §1.6

Problem 3.1
A mouse is wandering on a rectangular grid with grid points \((a, b), a, b = 0, 1, 2, 3, \ldots\). From grid position \((a, b)\), the mouse must move either to position \((a + 1, b)\) or \((a, b + 1)\).

(a) Find the total number of different paths from (0, 0) to \((n, n)\).

(b) Find the total number of different paths from (1, 0) to \((n + 1, n)\) that pass through at least one of the points \((r, r)\), with \(1 \leq r \leq n\).

Hint: Consider counting instead the number of unrestricted paths from (0, 1) to \((n + 1, n)\). By thinking about reflection across the diagonal, show that the number of such unrestricted paths is the same as the number of paths to count in part (b).

Problem 3.2
A lazy GSI returns exams to students randomly. If the class has \(n\) students, what is the probability that at least one student receives his/her own exam? Compute the limiting value of this probability as \(n \to +\infty\).

Hint: The inclusion-exclusion formula (p. 54, textbook) could be useful. You can also write a program to verify your expression via experimentation.

Problem 3.3
(Fishing trip) Koichi’s backyard pond contains \(g\) goldfish and \(c\) catfish. All fish are equally likely to be caught.

(a) Suppose that Koichi catches a total of \(k\) fish (no fish are thrown back). What is the probability of catching \(x\) goldfish?

(b) Now suppose that all \(k\) fish are returned to the pond, and he starts fishing again, and catches a total of \(m\) fish. What is the probability that among the caught set of \(m\) fish, exactly 2 goldfish are included that were also caught in the first catch? (Assume that fish do not learn from experience.)

Problem 3.4
In each running of a lottery, a sequence \(r\) numbers are drawn independently (with replacement) from a total of \(n\) numbers. You win the lottery if the \(r\)-number sequence on your ticket matches the drawn sequence.

(a) Suppose I buy one (randomly chosen) lottery ticket. What is the probability of winning the lottery?
(b) Suppose that I hack into the lottery computer, and program it to only draw sequences with \( r \) numbers in a non-decreasing order. I then go and buy a ticket with its numbers in non-decreasing order. Now what is my probability of winning the lottery?

**Hint:** Let \((x_1, x_2, \ldots, x_m)\) be a collection of integers with each \( x_i \geq 0 \) such that \( \sum_{i=1}^{m} x_i = k \). The number of such collections is \( \binom{m+k-1}{m-1} \).

**Problem 3.5**

An urn contains \( a \) aquamarine balls, and \( b \) blue balls. The balls are removed successively without replacement until the urn is empty. If \( b > a \), show that the probability that at all stages until the urn is empty there are more blue than aquamarine balls in the urn is equal to \( (b-a)/(b+a) \).

**Problem 3.6**

A communication system transmits one of three signals, \( s_1, s_2 \) and \( s_3 \), with equal probabilities. The transmission is corrupted by noise, causing the received signal to be changed according to the following table of conditional probabilities:

| Receive, \( j \) | \( P(s_j|s_i) \) |
|------------------|------------------|
|                  | \( s_1 \) | \( s_2 \) | \( s_3 \) |
| Send, \( i \)    | \( s_1 \) | 0.3    | 0.4    | 0.3    |
| \( s_2 \)        | 0.002  | 0.99   | 0.008  |
| \( s_3 \)        | 0.8    | 0.15   | 0.05   |

For example, if \( s_1 \) is sent, the probability of receiving \( s_3 \) is 0.3. The entries of the table list the probability of \( s_j \) received, given that \( s_i \) is sent, i.e., \( P(s_j \text{ received}|s_i \text{ sent}) \).

(a) Compute the (unconditional) probability that \( s_j \) is received for \( j = 1, 2, 3 \).

(b) Compute the probability \( P(s_i \text{ sent}|s_j \text{ received}) \) for \( i, j = 1, 2, 3 \).

(c) As seen from the numbers above, the transmitted signal can be very different from the received signal. Thus, when symbol \( s_j \) is received, it may not be a good idea to conclude that \( s_j \) was sent. We need a decision scheme to decide which signal was sent based on the signal we receive with the lowest possible error probability, that is, the probability of making a wrong decision. Find the scheme that minimizes the overall error probability.

**Problem 3.7**

Evelyn has \( n \) coins in her left pocket, and \( n \) coins in her right pocket. Every minute, she takes a coin out of a pocket chosen at randomly (and independently) from previous draws, until the first time one of her pockets is empty. For an integer \( t \in [1, n] \), compute the probability of having \( t \) coins in the other pocket when she stops.