Problem 1. Consider the random walk example in the notes, page 74. Derive the Kalman filter equations and implement it in Matlab to reproduce Figure 4.15.

Problem 2. Let $(X, Y)^T \sim \mathcal{N}([0; 0], [3, 1; 1, 1])$. Find $E[X^2 | Y]$.

Problem 3. Let $(X, Y, Z)^T \sim \mathcal{N}([0; 0; 0], [5, 3, 1; 3, 9, 3; 1, 3, 1])$. Find $E[X | Y, Z]$.

Problem 4. Consider arbitrary random variables $X$ and $Y$. Prove the following property.

\[
\text{Var}(Y) = E(\text{Var}(Y | X)) + \text{Var}(E(Y | X)).
\]

Problem 5. Let the joint density of two random variables $X$ and $Y$ be

\[
f_{X,Y}(x,y) = \frac{1}{4}(2x + y)1\{0 \leq x \leq 1\}1\{0 \leq y \leq 2\}.
\]

First show that this is a valid joint distribution. Suppose you observe $Y$ drawn from this joint density. Find MMSE$[X | Y]$.

Problem 6. Given four independent $N(0, 1)$ random variables $X$, $Y$, $Z$, and $V$, find the following minimum mean square estimator:

\[
E[X + 2Y + 3Z | Y + 5Z + 4V]
\]

Find the mean squared error of the estimator.