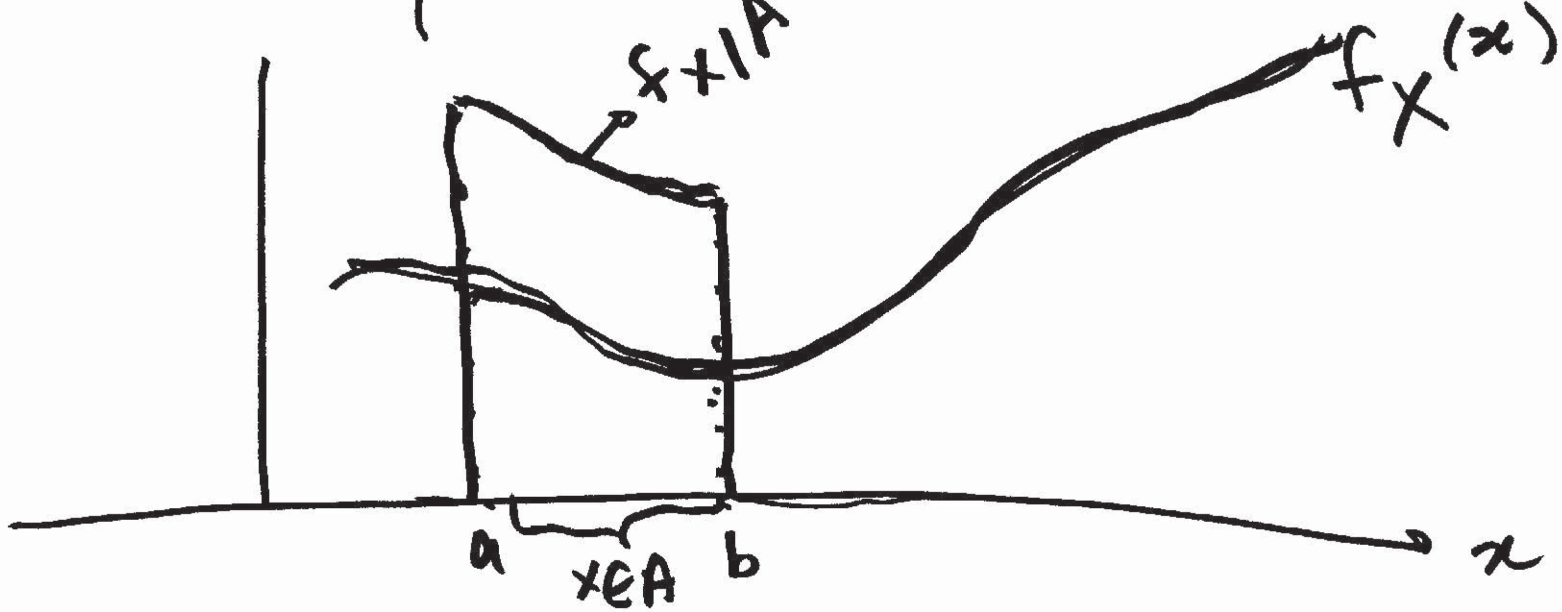
Conditioning on an Event

Cont. R.V. X.
given an event $\{x \in A\}$

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(x \in A)} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$



$$P(X \in B \mid X \in A) = \frac{P(X \in B , X \in A)}{P(X \in A)}$$

$$= \frac{\int_{A \cap B} f_X(x)\, dx}{P(X \in A)} \quad (*)$$

$$= \int_B f_{X|A}(x)\, dx$$

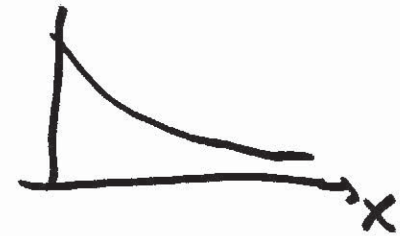
More generally:

$$P(X \in B / A) = \int_B f_{X|A}(x)\, dx$$

EX Exponential R.V. is memoryless.

T = R.V. denotes Time until a bulb burns out.

Pdf: $f_T(x) = \lambda e^{-\lambda x}$ $x > 0$



CDF: $P_r(T < x) = F(x) = 1 - e^{-\lambda x}$ $x > 0$

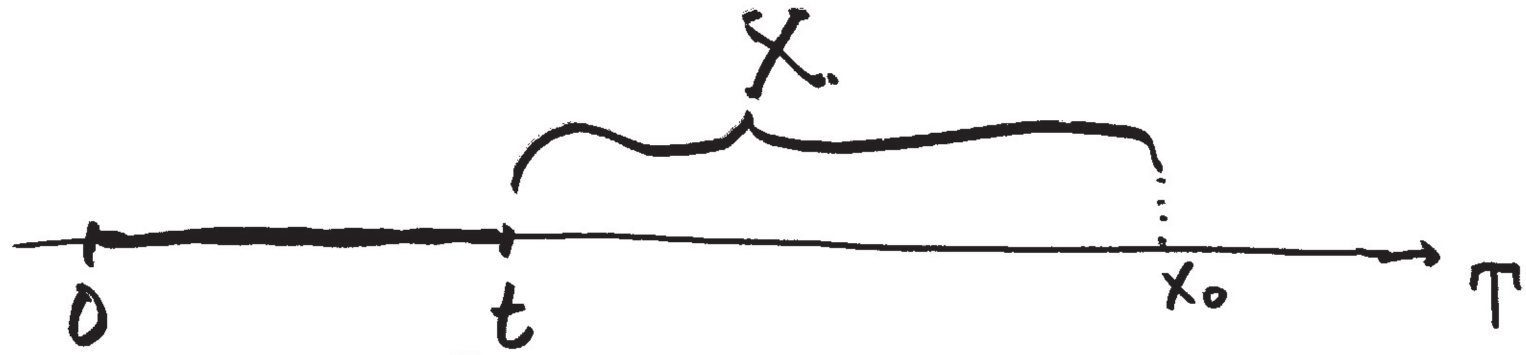
- Turn light on, leave, return in t seconds later, observe light is still on.

$$A = \{T > t\}$$

Q Compute Pdf of the remainder of the life

X = additional Time until lightbulb burns out

$$P(X > x_0 \mid A) = P(T > t + x_0 \mid A)$$



Time of return

$$= P(T > t + x_0 \mid T > t)$$

$$= \frac{P(T > t + x_0 , T > t)}{P(T > t)}$$

$$P(X > x_0 \mid A) = \frac{P(T > t + x_0)}{P(T > t)}$$

$$P(X > x_0 | A) = \frac{1 - (1 - e^{-\lambda(t+x_0)})}{1 - (1 - e^{-\lambda t})}$$

$$P(X > x_0 | A) = \frac{e^{-\lambda(t+x_0)}}{e^{-\lambda t}} = e^{-\lambda x_0}$$

$$\boxed{P(X > x_0 | A) = e^{-\lambda x_0}}$$ → Amazing.

<u>Conclusion</u> Conditional CDF of X is independent of t ie time of return ⟹ memoryless

Conditional Expectation

$$E(X/A) = \int_{-\infty}^{+\infty} x\, f_{X/A}(x)\, dx$$

$$E(g(x)/A) = \int_{-\infty}^{+\infty} g(x)\, f_{X/A}(x)\, dx$$

- $A_1\ A_2 \ldots\ A_n$ disjoint form a partition.

$$f_X(x) = \sum_{i=1}^{n} p(A_i)\, f_{X|A_i}(x)$$

Total Exp. thm:

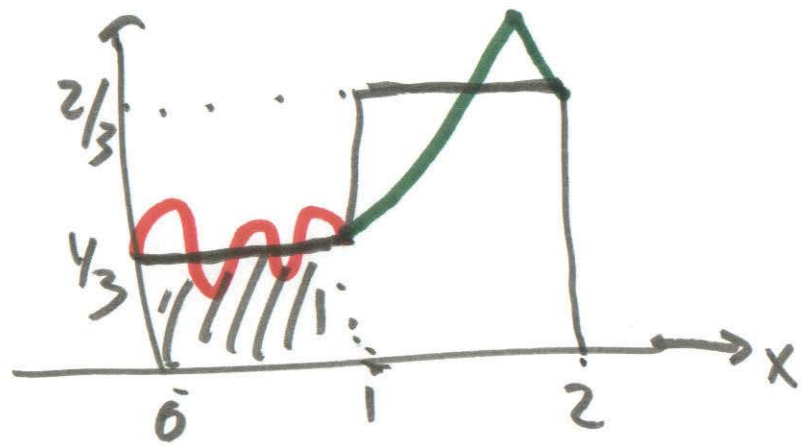
$$E(X) = \sum_{i=1}^{n} P(A_i)\, E(X|A_i)$$

Total Expectation thm

$$E(g(x)) = \sum_{i=1}^{n} P(A_i)\, E[g(x) \mid A_i]$$

EX R.V. X

$$f_X(x) = \begin{cases} 1/3 & 0 \le x \le 1 \\ 2/3 & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$



$A_1 = \{x \in [0,1]\}$

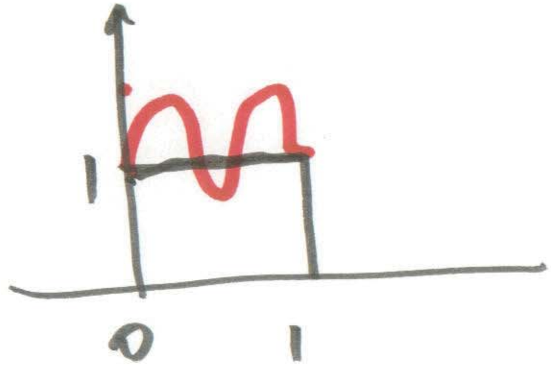
$A_2 = \{x \in [1,2]\}$

Q conditional mean, 2nd moment of X.

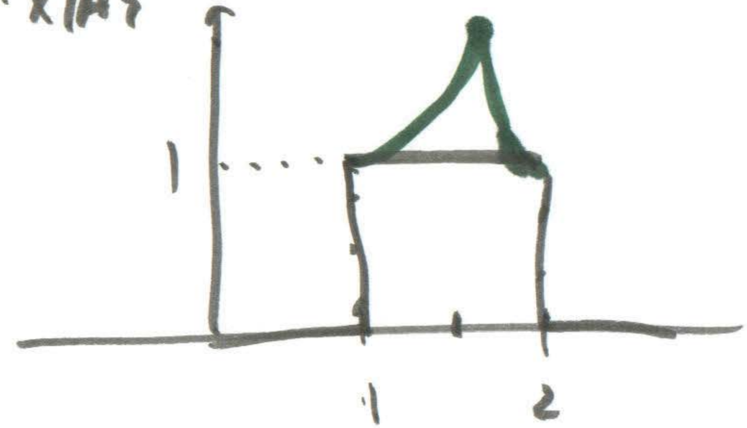
$$E(X \mid A_1) \quad E(X \mid A_2) \quad E(X^2 \mid A_i)$$

$$P(A_1) = \int_0^1 f_X(x)\,dx = \frac{1}{3} \qquad P(A_2) = \int_1^2 f_X(x)\,dx = 2/3$$

$f_{X|A_1}(x)$



$f_{X|A_2}(x)$



$$E[X|A_1] = \frac{1}{2} \qquad E[X/A_2] = 1.5$$

$$E[X^2/A_1] = \frac{1}{3} \qquad E[X^2/A_2] = \frac{7}{3}$$

$$E[X] = P(A_1)E[X|A_1] + P(A_2)E[X|A_2]$$

$$= \frac{1}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{3}{2} = 7/6$$

$$E[X^2] = P(A_1)\,E[X^2|A_1] + P(A_2)E[X^2/A_2] =$$

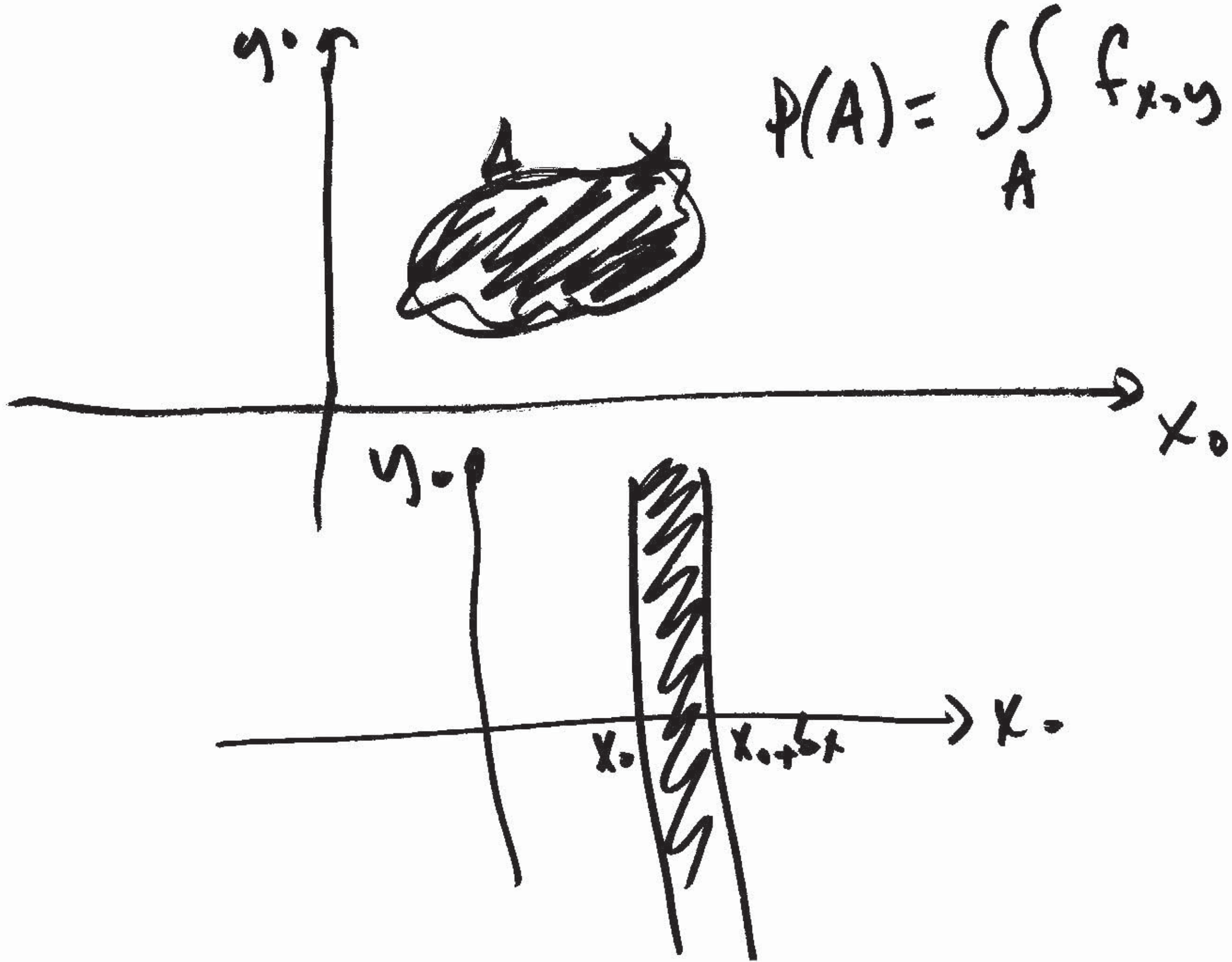
$$= \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{7}{3} = 15/9$$

Multiple Cont. R.V.

2D event space. for possible exp. values of RV. X, Y.

$f_{X,Y}(x_0, y_0)$ $\qquad$ $f_{X,Y}(x, y)$

$$P(A) = \iint_A f_{X,Y}(x_0, y_0)\,dx_0\,dy_0$$



$$\Pr(x_0 < x < x_0 + \delta x)$$

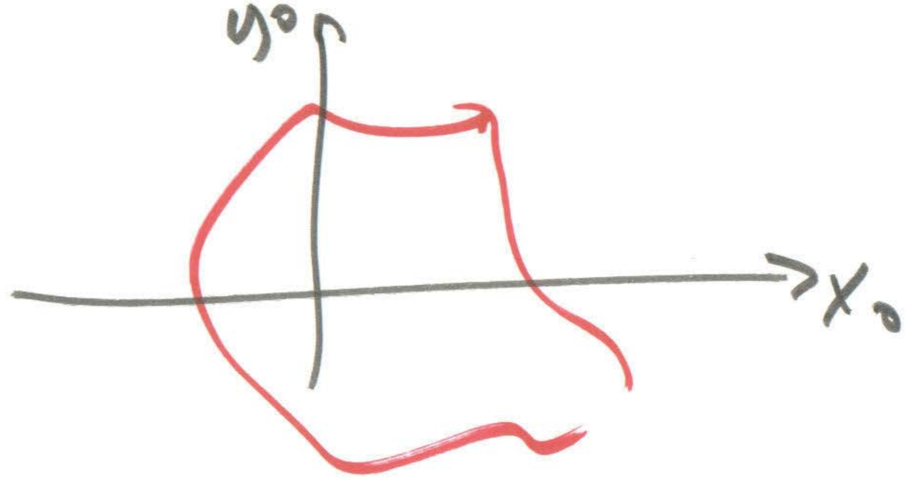
$$= \int_{y_0=-\infty}^{+\infty} f_{x,y}(x_0, y_0)\, dx_0\, dy_0$$

$$= dx_0 \int_{-\infty}^{+\infty} f_{x,y}(x_0, y_0)\, dy_0$$

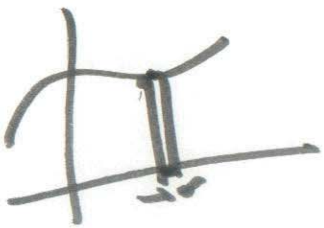
$$= dx_0 \quad f_x(x_0)$$

$$\Longrightarrow \boxed{\begin{aligned} f_x(x_0) &= \int_{y_0=-\infty}^{+\infty} f_{x,y}(x_0, y_0)\, dy_0 \\ f_y(y_0) &= \int_{x_0=-\infty}^{+\infty} f_{x,y}(x_0, y_0)\, dx_0 \end{aligned}}$$

$$\iint_{x_0 y_0} f_{x,y}(x_0,y_0)\,dx_0\,dy_0 = 1$$



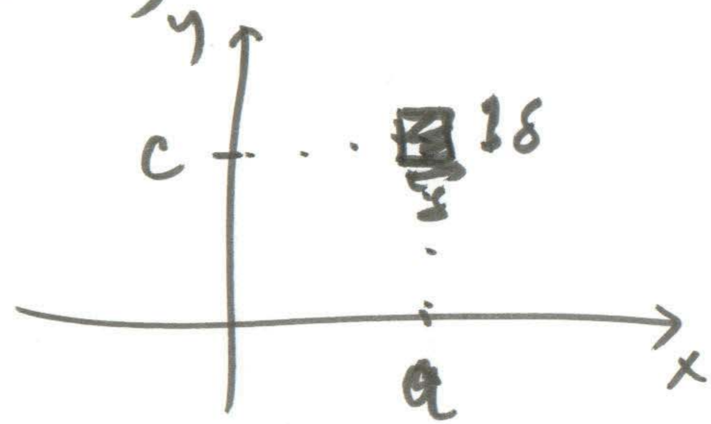
$$f_{x,y}(x_0,y_0) \geq 0$$



$$P(\, a < x < a+\delta \,,\; c < Y < c+\delta \,)$$

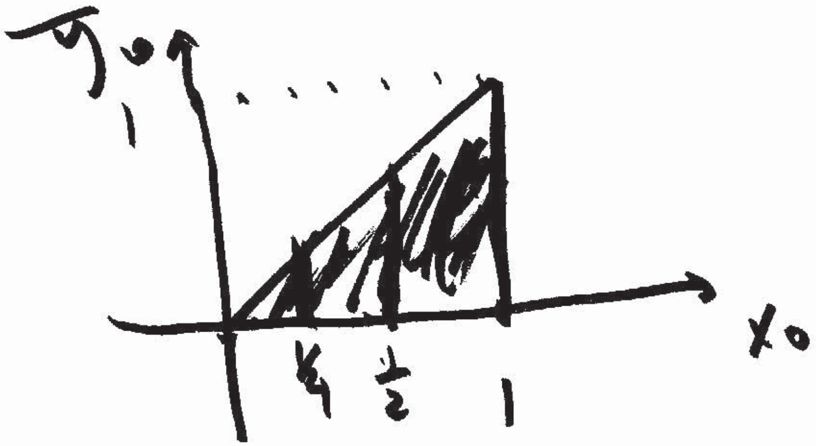
$$= \int_c^{c+\delta} \int_a^{a+\delta} f_{X,Y}(x,y)\,dx\,dy$$

$$\approx \delta^2 f_{X,Y}(a,c)$$



EX $f_{X,Y}(x_0, y_0) = \begin{cases} A x_0 & 0 \le y_0 \le x_0 \le 1 \\ 0 & \text{otherwise.} \end{cases}$

① Find A, Find $f_X(x_0)$ find $P(XY < 0.25)$

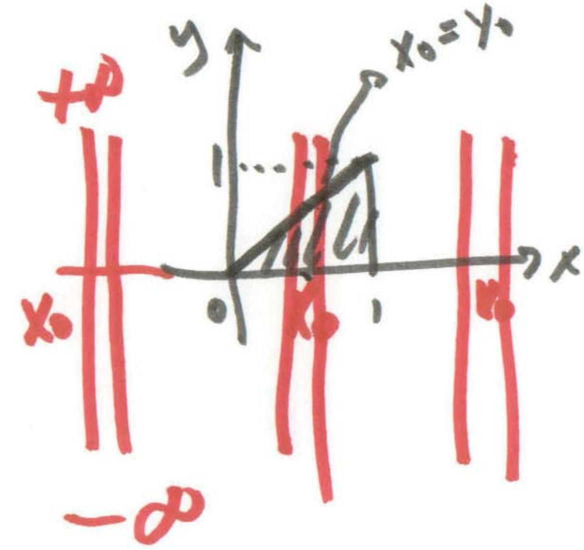


$$\int_{x_0=-\infty}^{+\infty} \int_{y_0=-\infty}^{+\infty} f_{X,Y}(x_0, y_0) = 1$$

$$= \int_{x_0=0}^{1} A x_0^2 \, dx_0 = 1$$

$$\Rightarrow \frac{A}{3} = 1 \longrightarrow \boxed{A = 3}$$

$$f_X(x_0) = \int_{-\infty}^{+\infty} f_{X,Y}(x_0, y_0)\, dy_0$$

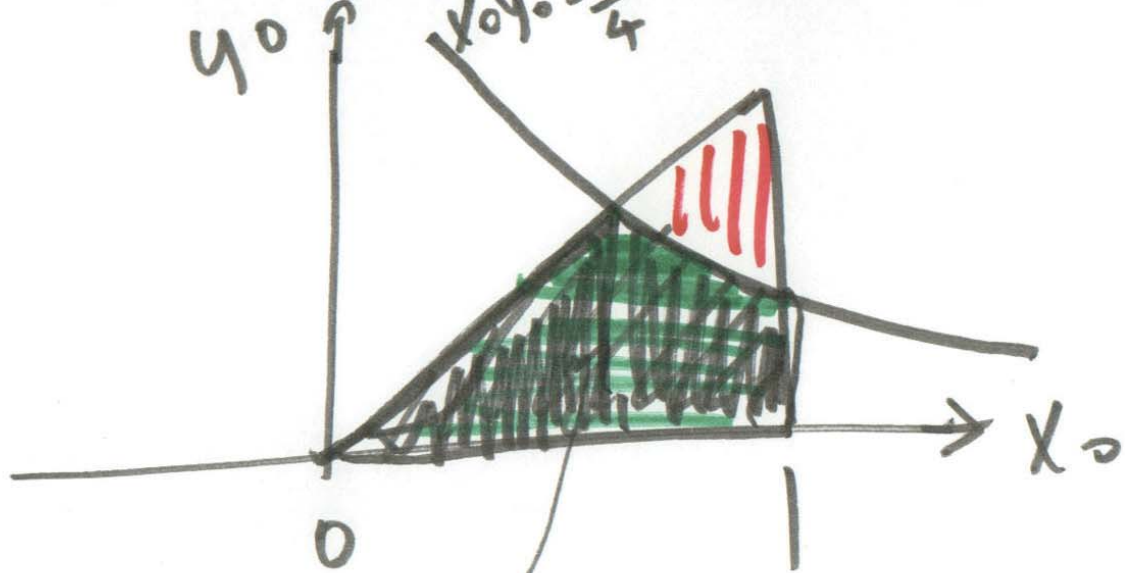


$$= \begin{cases} \displaystyle\int_{y_0=0}^{x_0} 3x_0\, dy_0 & 0 \le x_0 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_X(x_0) = \begin{cases} 3x_0^2 & 0 \le x_0 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{x_0=0}^{1} 3x_0^2\, dx_0 = 1 \quad \left[\frac{x_0^3}{3}\right]_0^1$$

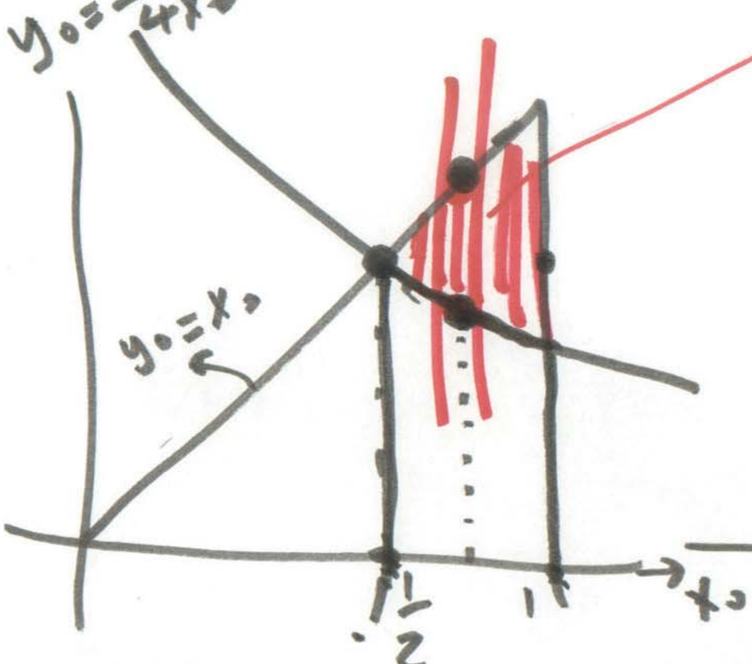
$P(XY < 0.25)$

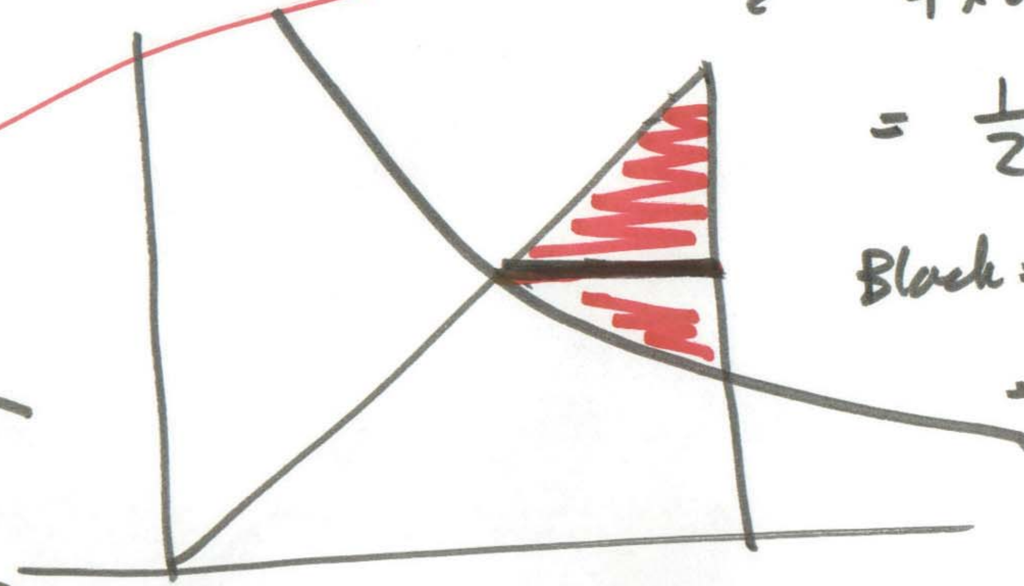


$= 1 - \Pr(\text{red region}).$

$$\int_{x_0=\frac{1}{2}}^{1}\int_{y_0=\frac{1}{4x_0}}^{x_0} f(x_0,y_0)\, dx_0\, dy_0$$

$= \frac{1}{2}$





Black $= 1 - \frac{1}{2} = \frac{1}{2}$

$= \Pr(xy < \frac{1}{4})$