Expectation for Multiple Cont. R.V.

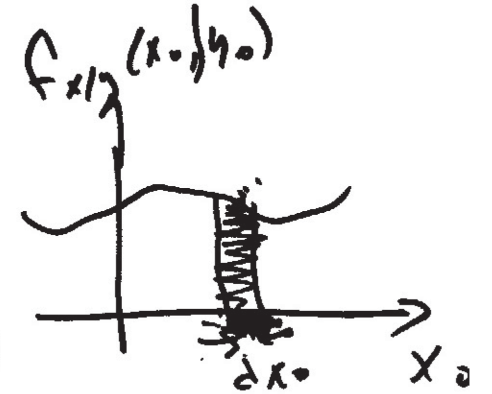
- $X, Y \longrightarrow f_{X,Y}(x_0, y_0)$

fn of x, y $\quad g(x,y)$

Q : $E[g(x,y)] = \int\int_{-\infty}^{+\infty} g(x_0, y_0)\, f_{X,Y}(x_0, y_0)\, dx_0\, dy_0$

- X, Y

$E[aX + bY] = a\, E[X] + b\, E[Y]$



- Conditional pfd : $f_{X|Y}(x_0|y_0)$

Conditional prob that $x_0 < X < x_0 + dx_0$

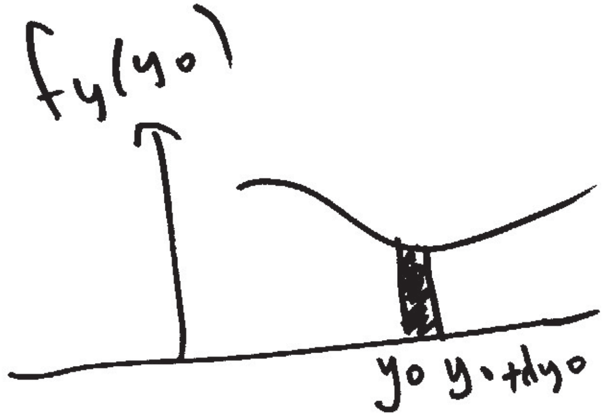
given. $y_0 < Y < y_0 + dy_0$

Event A $\quad x_0 \le X < x_0 + dx_0$

B $\quad y_0 \le y \le y_0 + dy_0$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{f_{x,y}(x_0,y_0)\,dx_0\,dy_0}{f_y(y_0)\,dy_0}$$

$$= \frac{f_{x,y}(x_0,y_0)}{f_y(y_0)}\,dx_0$$

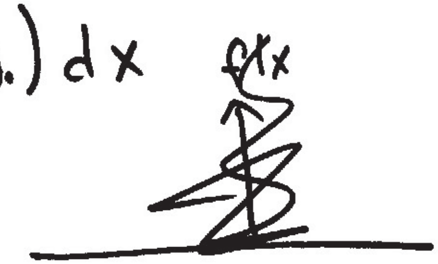


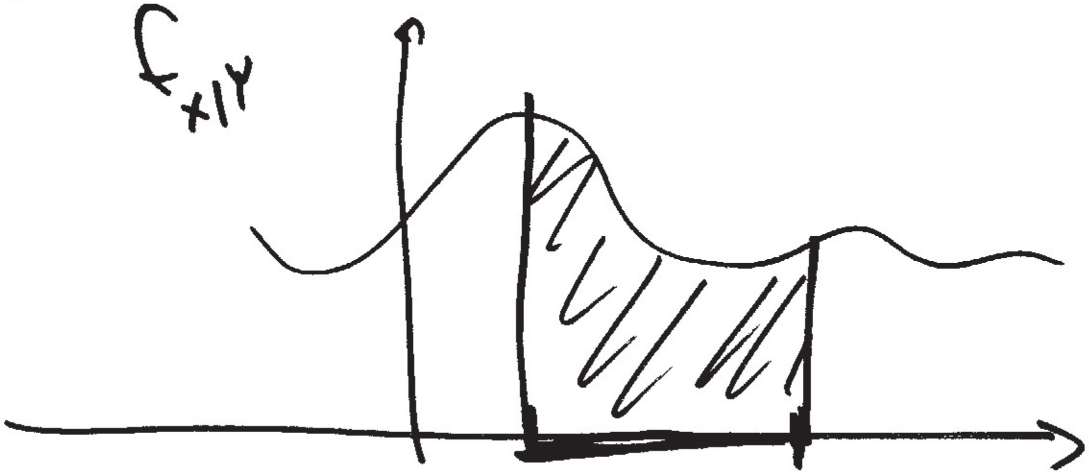
$$\Rightarrow \boxed{f_{x|y}(x_0|y_0) = \frac{f_{x,y}(x_0,y_0)}{f_y(y_0)}}$$

Prop: $\int_{-\infty}^{\infty} f_{x|y}(x_0|y_0)\,dx_0 = 1.$

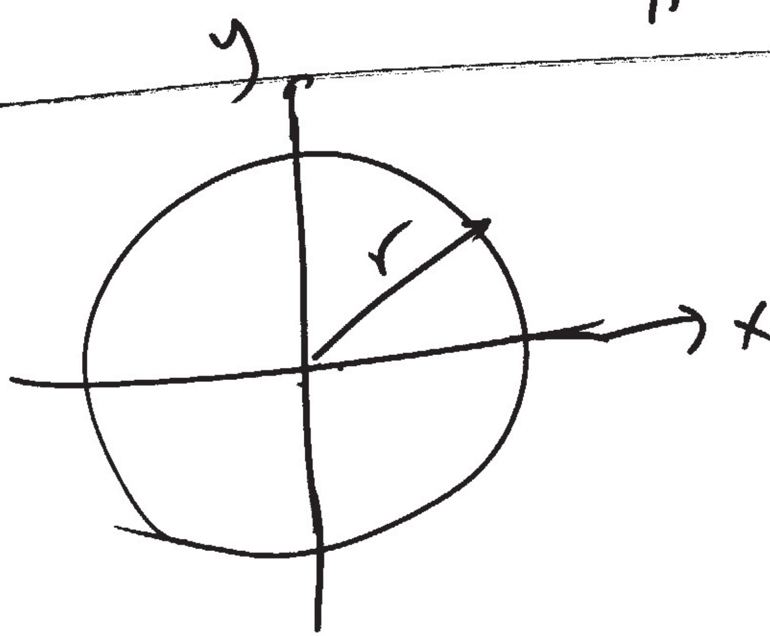
$$- \; f_{x,y|A}(x_0,y_0|A) = \begin{cases} \dfrac{f_{x,y}(x_0,y_0)}{P(A)} & x_0,y_0 \in A \\ 0 & \text{otherwise.} \end{cases}$$

$P(X \in A \mid Y = y) = \int_A f_{X|Y}(x|y_0)\,dx$





Ex



$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2+y^2<r^2 \\ 0 & \text{otherwise} \end{cases}$$

① $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$$

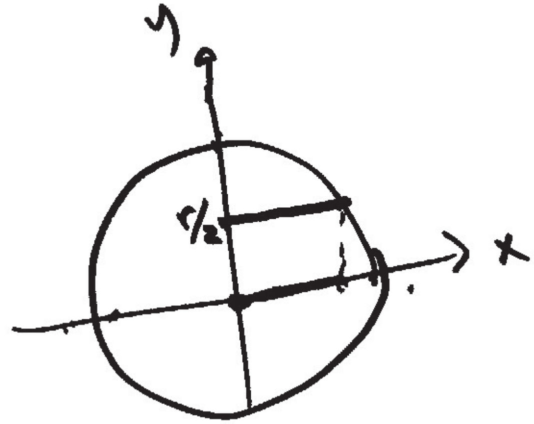
$$= \begin{cases} \frac{1}{\pi r^2} \int_{x^2+y^2<r^2} 1 \, dx \end{cases}$$

$$= \begin{cases} \frac{1}{\pi r^2} \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} dx \end{cases}$$

$\vdots$

$$f_Y(y) = \begin{cases} \frac{2}{\pi r^2} \sqrt{r^2 - y^2} & \text{if } |y| < r \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X|Y} = \begin{cases} \dfrac{\frac{1}{\pi r^2}}{\frac{2}{\pi r^2}\sqrt{r^2-y^2}} & x^2+y^2<r^2 \\ 0 & \text{otherwise} \end{cases}$$

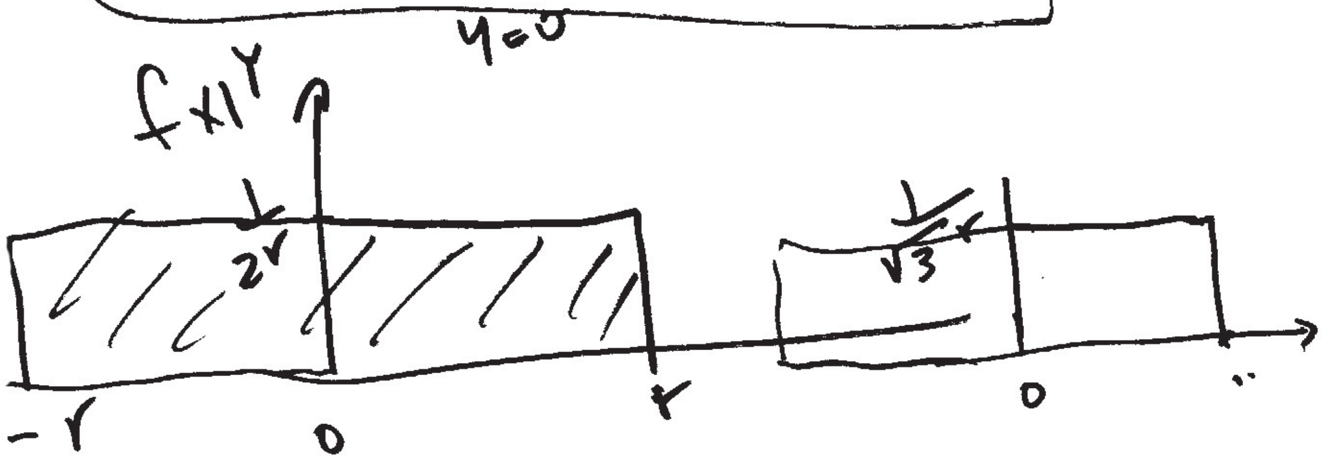


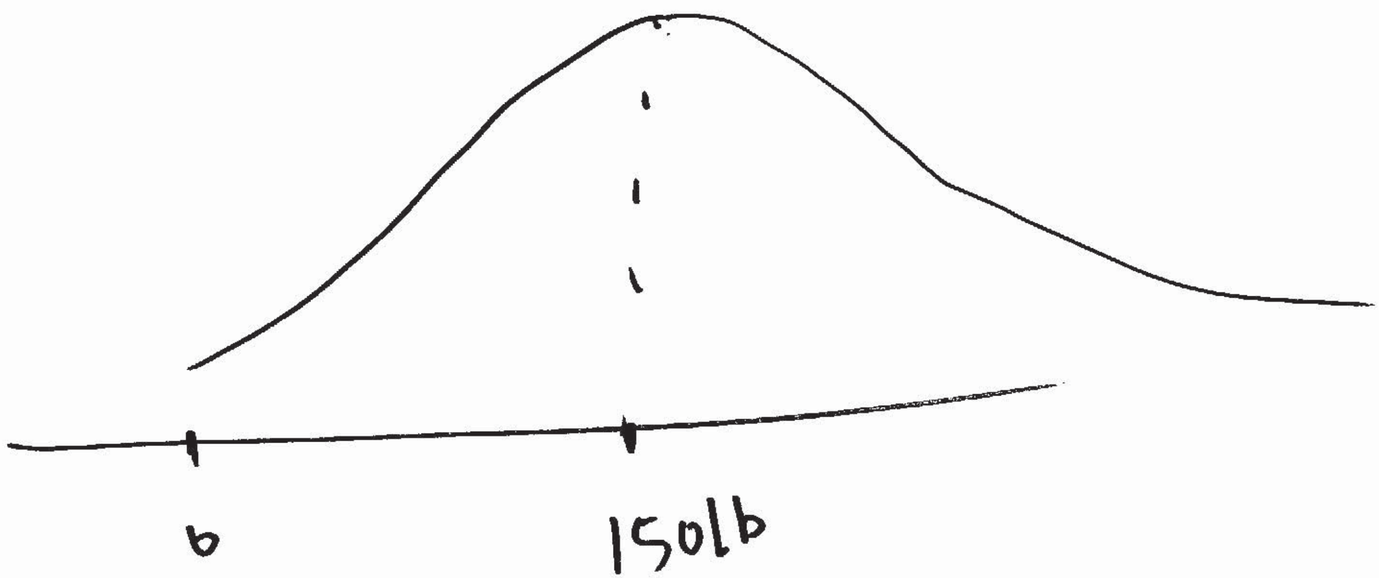
$$\boxed{f_{X|Y}(x|y) = \frac{1}{2\sqrt{r^2-y^2}}}$$

$y = 0$

$f_{X|Y} = \frac{1}{2r}$

$y = \frac{r}{2}$ $f_{X|Y} = \frac{1}{2\sqrt{\frac{3r^2}{4}}} = \frac{1}{\sqrt{3}\, r}$

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Facts About Multiple R.V. .

$$f_X(x) = \int_{-\infty}^{+\infty} f_Y(y)\, f_{X|Y}(x|y)\, dy$$

Expectation:

$$E[g(X,Y)] = \iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$E[g(X,Y) \mid Y=y] = \int g(x,y)\, f_{X|Y}(x|y)\, dx.$$

Total Expectation Thm:

$$E(X) = \int E[X \mid Y=y]\, f_Y(y)\, dy$$

$$E[g(X)] = \int E[g(X \mid Y=y)]\, f_Y(y)\, dy$$

Independence of Cont. R.V.

2 continous R.V. ~~X~~, Y are indp. iff.

$f_{X|Y}(x_0, y_0) = f_X(x_0)$ $\forall x_0, y_0$

equivalently: $f_{X,Y}(x_0, y_0) = f_X(x_0)\, f_Y(y_0)$

X, Y independent . $E[XY] = E[X]\, E[Y]$.

$E[g(X)h(Y)] = E[g(X)]\, E[h(Y)]$.

$Var(X+Y) = Var(X) + Var(Y)$

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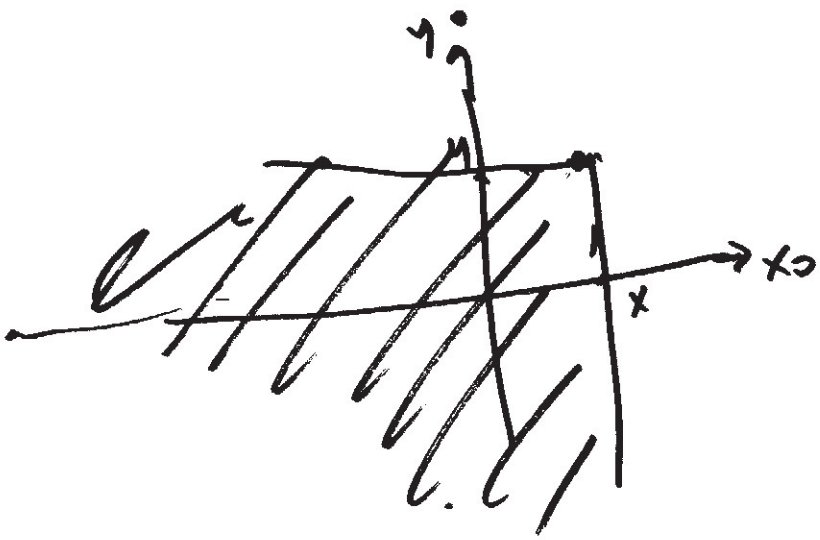
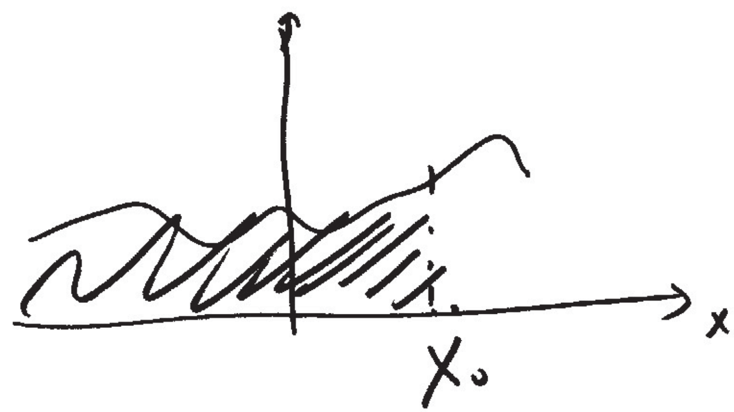
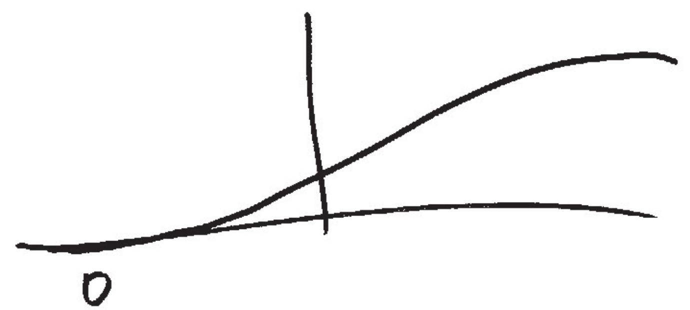
joint CDF ..

$$F_{X,Y}(x,y) = P(X < x, Y < y)$$

$$= \int_{s=-\infty}^{x} \int_{t=-\infty}^{y} f_{X,Y}(s,t)\, ds\, dt$$



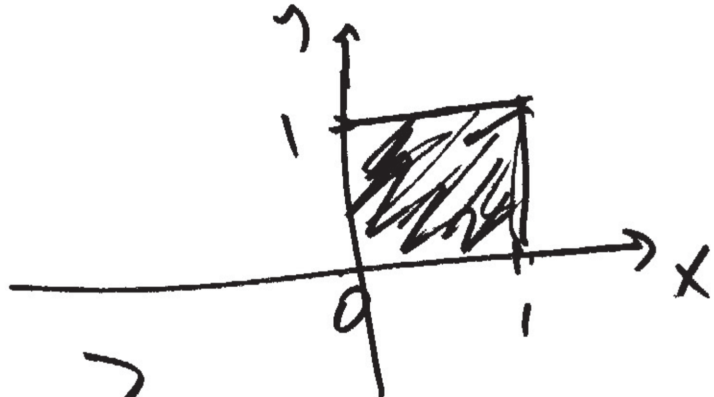




$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} = f_{X,Y}(x,y)$$

Ex $F_{X,Y}(x,y) = xy \quad 0 \le x,y \le 1$

what is pdf? $f_{X,Y}(x,y) = 1 \quad 0 \le x,y \le 1$



$$\frac{\partial}{\partial y}\left\{ \frac{\partial F}{\partial x}\{xy\} = y \right\} = 1$$

Bayes Rule in Continous domain

- X as a R.V. pdf $\boxed{f_X}$
- measure X get new R.V. Y

  Y noisy version of X. based on measurement process.

  - we know pdf $\boxed{f_{Y|X}}$
- given Y, what is the pdf of X $f_{X|Y}$?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{\int \underline{f_X(t)}\, f_{Y/X}(y|t)\, dt}$$

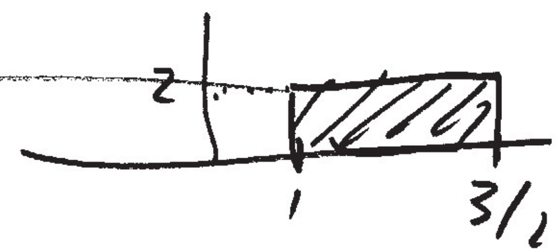
EX lightbulb life. exponential Y.

λ is also random. uniformly distributed

$[1, 3/2]$

Q what is $f_{\lambda|Y}$

$f_\Lambda(\lambda) = 2 \qquad 1 \le \lambda \le 3/2$



$$f_{\Lambda|Y} = \frac{f_{\Lambda}(\lambda)\, f_{Y|\Lambda}(y/\lambda)}{\int f_{\Lambda}(t)\, f_{Y|\Lambda}(y/t)\, dt}$$

$$f_{\Lambda|Y}(\lambda|y) = \frac{2\lambda e^{-\lambda y}}{\int_1^{3/2} 2\, t e^{-ty}\, dt} = \cdots$$

~~Compute $P(A \mid Y=y)$~~

Variation of Bayes

Event A.

$P(A)$

$Y =$ Cont. R.V.

$f_{Y/A}(y)$ known.

$f_{Y/A^c}(y)$ known.

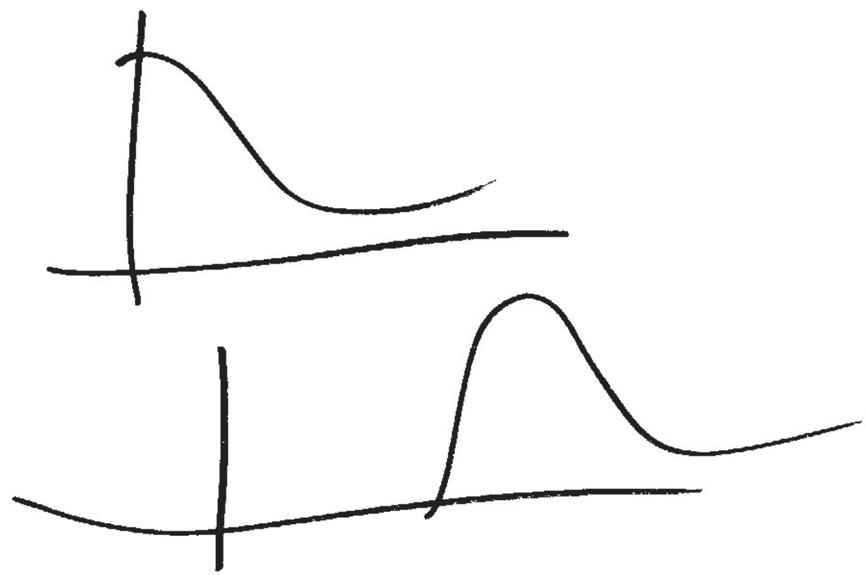
Compute $P(A / Y=y)$

Ex rain Event A

y = distribution of pollen in the air.

if rains

if not rain



$$P(A \mid Y=y) = \frac{P(A)\, f_{Y|A}(y)}{P(A)\, f_{Y|A}(y) + P(A^c)\, f_{Y|A^c}(y)}$$

Variation

Event A can take one of $\{N=n\}$ values

$N =$ discrete R.V. $\quad P_N(n)$

$Y =$ cont. R.V. $\quad f_{Y|N}(y|n)$

$$P(N=n \mid Y=y) = \frac{P_N(n)\, f_{Y|N}(y|n)}{f_Y(y)}$$

$$f_Y(y) = \sum_i P_N(i)\, f_{Y|N}(y|i)$$

$$P(N=n \mid Y=y) = \frac{P_N(n)\, f_{Y|N}(y|n)}{\sum_i P_N(i)\, f_{Y|N}(y|i)}$$

Ex Binary signal $S = \pm 1$

$P(S=1) = p$ $\quad P(S=-1) = 1-p$

Receiver: $Y = N + S$

noise

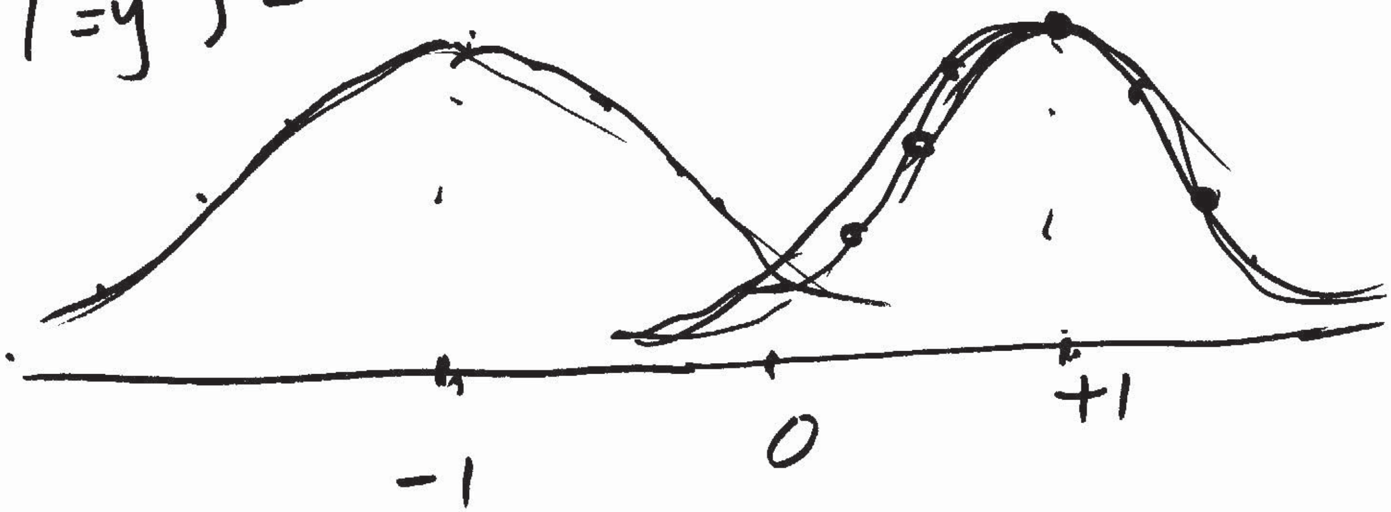
signal.

$N \sim N(0,1)$ normally mean 0 var 1

normal mean of 1 var 1 $y = N+1$

normal

$P(S=1 \mid Y=y) =$



$$= \frac{P(S=1)\, f_{Y|S}(y \mid 1)}{f_Y(y)}$$

$$= \frac{P\, e^{-\frac{(y-1)^2}{2}}}{P(S=1)\, f_{Y/N}(y \mid 1) + P(S=-1)\, f_{Y/N}(y \mid -1)}$$

$$P(S=1 \mid Y=y) = \frac{p\, e^{-\frac{(y-1)^2}{2}}}{p\, e^{-\frac{(y-1)^2}{2}} + (1-p)\, e^{-\frac{(y+1)^2}{2}}}$$

$y = S + N$

-1

$s=-1$ $\quad y = N - 1$

normal mean -1 var 1

$$P(S=-1 \mid Y=y)$$