Derived PDF:

R.V. X. → known pdf.

$y = g(x)$

Compute pdf of $g(x)$ in Terms of pdf of X.
y

2 step process.

① Find prob event $g \leq g_0$ $\forall g_0$

→ CDF for g.

② Differentiate CDF in step 1 to get pdf. ⇒ Differentiate w.r.t g_0

To get $f_g(g_0)$

Step ① : $Y = g(X)$.

Find $F_Y(y) = P(g(X) < y)$

CDF

$$= \int_{\{x \mid g(x) < y\}} f_X(x)\, dx.$$

Step ② : $f_Y(y) = \dfrac{d F_Y(y)}{dy}$

pdf

Ex wheel of Fortune. $0.00 \longrightarrow 1.00$

spin twice.

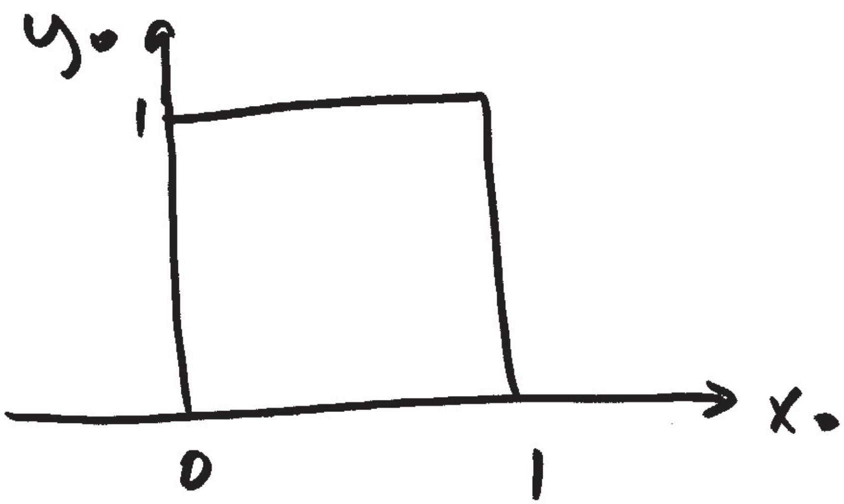
$X =$ first reading

$y =$ 2nd reading.

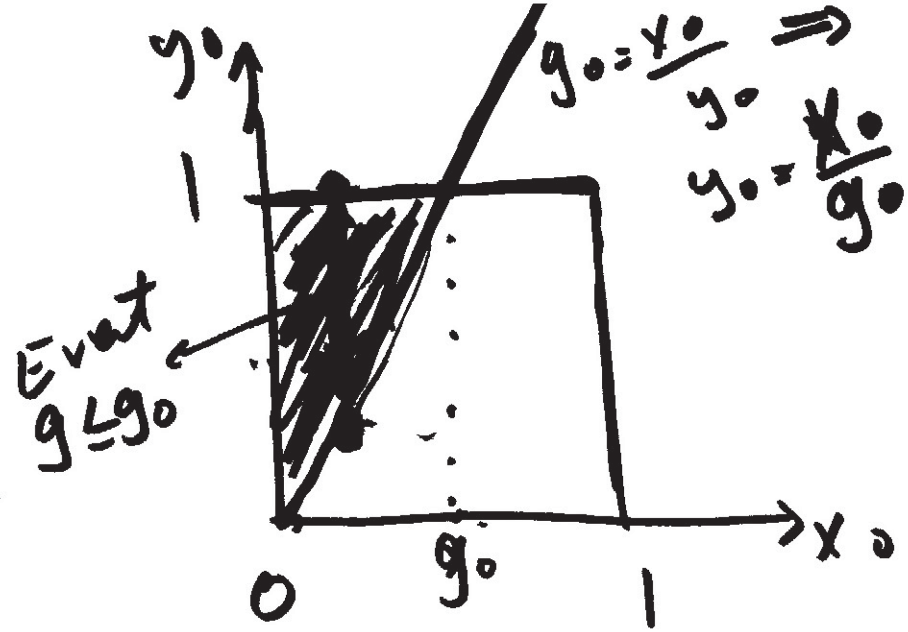
2 spins are independent.

Q pdf $g(x,y) = \dfrac{x}{y}$ compute $f_g(g_0)$

$$f_{X,Y}(x_0,y_0) = f_X(x_0)\, f_Y(y_0) = \begin{cases} 1 & 0 < x_0 \leq 1,\ 0 \leq y_0 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$P_{g_{\leq}}(g_0)$ ..



Case ① $0 \le g_0 \le 1$

Case ② $1 \le g_0 \le \infty$

$$\text{Case ① : } P_{g_\le}(g_0) = \int_0^{g_0} dx_0 \int_{y_0 = \frac{x_0}{g_0}}^{1} 1 \, dy_0$$

$$\boxed{P_{g_\le}(g_0) = \frac{g_0}{2}}$$ ← CDF

Case (2): $1 \leq g_0 \leq \infty$

$$P_{g \leq g_0} = 1 - \text{unshaded area} = \text{shaded area.}$$

$$= 1 - \int_{x_0=0}^{1} dx_0 \int_0^{\frac{g_0}{x_0}} 1 \, dy_0$$

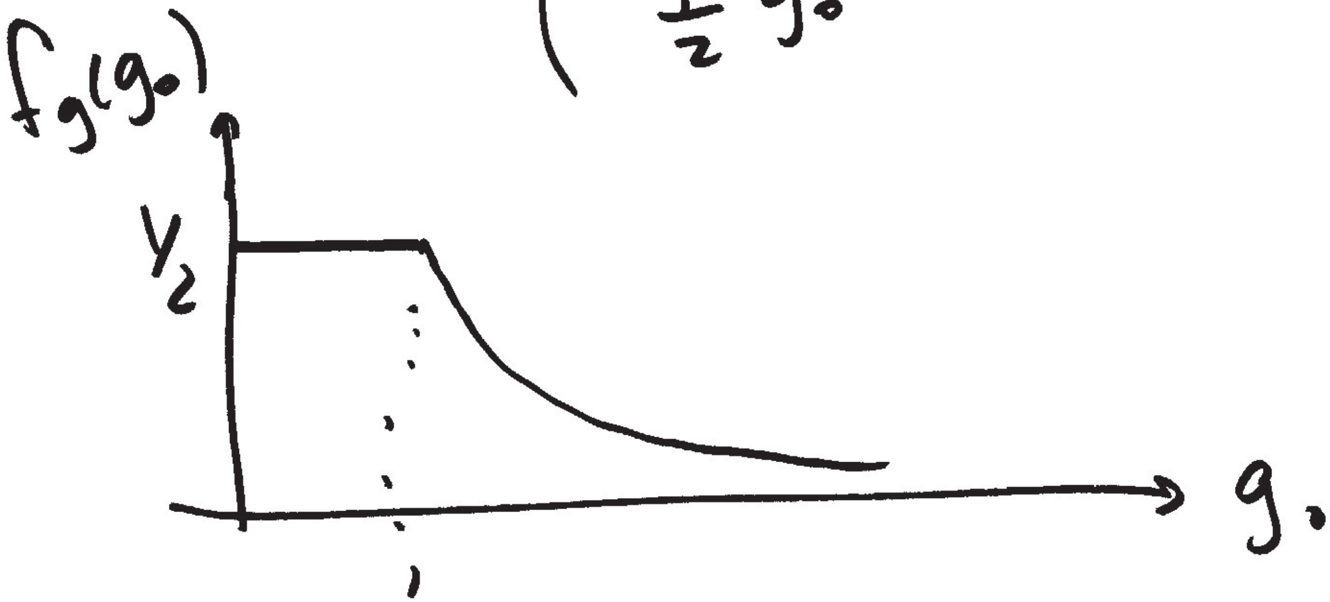
$$\boxed{P_{g_\leq}(g_0) = 1 - \frac{1}{2g_0}} \Longleftarrow \text{CDF.}$$

$$P_{g_\leq}(g_0) = \begin{cases} 0 & g_0 \leq 0 \\ \frac{g_0}{2} & 0 < g_0 \leq 1 \\ 1 - \frac{1}{2g_0} & 1 \leq g_0 \leq \infty \end{cases}$$

CDF

Differentiate to get Pdf:

$$f_g(g_0) = \begin{cases} 0 & g_0 \le 0 \\ \frac{1}{2} & 0 \le g_0 \le 1 \\ \frac{1}{2} g_0^{-2} & 1 \le g_0 \le \infty \end{cases}$$



Ex $\quad y = g(x) = x^2$

x is a ~~know~~ cont. R.V. with known pdf.

Q what is pdf for y?

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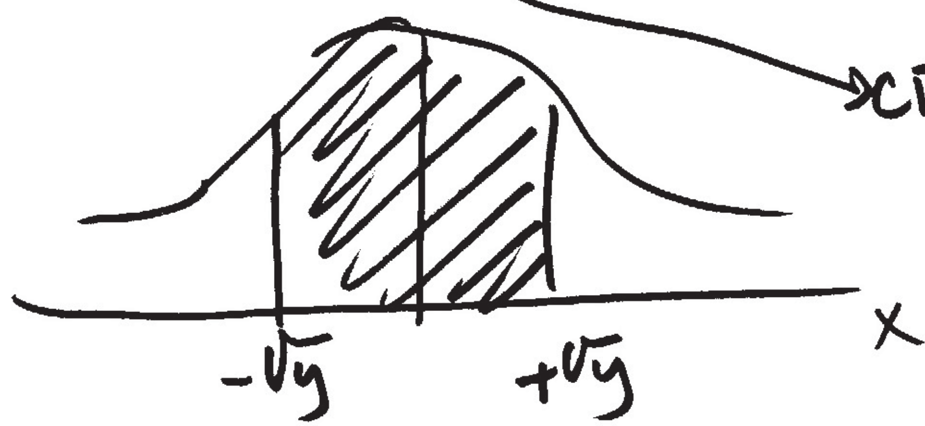
① Find $F_Y(y) = P(g(x) < y) = \int_{\{x \mid g(x) < y\}} f_X(x)\,dx$

$$F_Y(y) = P(Y < y)$$
$$= P(X^2 < y)$$
$$= P(-\sqrt{y} \le X < \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

CDF for x



$$\frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

pdf for x

Pdf of a Linear fn of a Cont. R.V.

X is cont. R.V. Pdf f_X → known.

$Y = aX + b$ $a \neq 0$ a, b constant.

Q what is $\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$

Proof assume $a > 0$ without loss of generality

CDF for y

$$\leftarrow F_Y(y) = P(Y \le y)$$

$$= P(aX + b \le y)$$

$$= P\left(X < \frac{y-b}{a}\right) \quad a > 0$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

Now Diff to get pdf.

$$\frac{dF_Y(y)}{dy} = \frac{d}{dy}\left\{F_X\left(\frac{y-b}{a}\right)\right\} \leftarrow \text{chain rule.}$$

$$\boxed{f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a > 0}$$

Show similarly for $a < 0$

EX X is normal mean μ, var σ^2

$Y = aX + b$ $a \neq 0$ what is pdf Y?

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}}$$

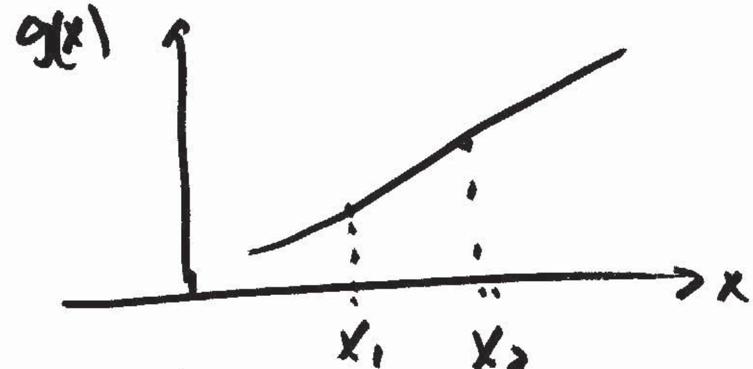
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,|a|\,\sigma} e^{-\frac{(y-b-a\mu)^2}{2\sigma^2 a^2}}$$

normal with mean $a\mu + b$, var $a^2\sigma^2$

PDF for a strictly monotic fn of a cont. R.V. X.

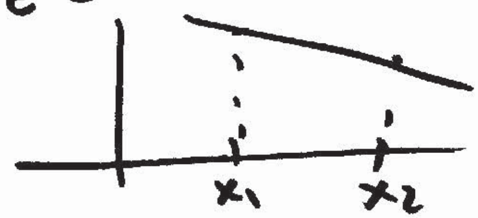
Def strictly monotically increasing fn g over an interval I is defined.

$$g(x) < g(x') \quad \forall x, x' \in I \quad \text{s.th} \quad x < x'$$



Def monotonically decreasing. fn g over an interval I is defined as:

$$g(x) > g(x') \quad \forall x, x' \in I \quad \text{s.th} \quad x < x'$$



Factoid: if g strictly monotonic $\exists h$

i.e. inverse. s.t. $x \in I$

$$y = g(x) \iff x = h(y)$$

$$\begin{tikzcd} x \arrow[r, bend left] & g(x) = y \arrow[l, bend left, "\begin{matrix} h(g(x)) \\ h(y) \end{matrix}"] \end{tikzcd}$$

Ex $g(x) = 5/x$

$$\begin{tikzcd} x \arrow[r, bend left] & \begin{matrix} g(x) \\ \shortparallel \\ 5/x = y \end{matrix} \arrow[l, bend left, "h(y) = 5/y"] \end{tikzcd}$$

$$h(5/x) = \frac{5}{5/x} = x$$

Ex $g(x) = ax + b \to h(y) = \frac{y - b}{a}$

$$\begin{tikzcd} x \arrow[r, bend left, "g(x)"] & y = g(x) = ax + b \arrow[l, bend left, "h(y) = \frac{y-b}{a}"] \end{tikzcd}$$

$$\begin{aligned} x &= h(ax+b) \\ &= \frac{(ax+b)-b}{a} \\ &= \frac{ax}{a} = x \end{aligned}$$

PDF for strictly monotonic fn of a Cont. R.V. X

R.V. X, Known pdf. f_X.

g strictly monotonic. with inverse h over range of X

$$y = g(x) \Longleftrightarrow x = h(y)$$

Assume h is differentiable.

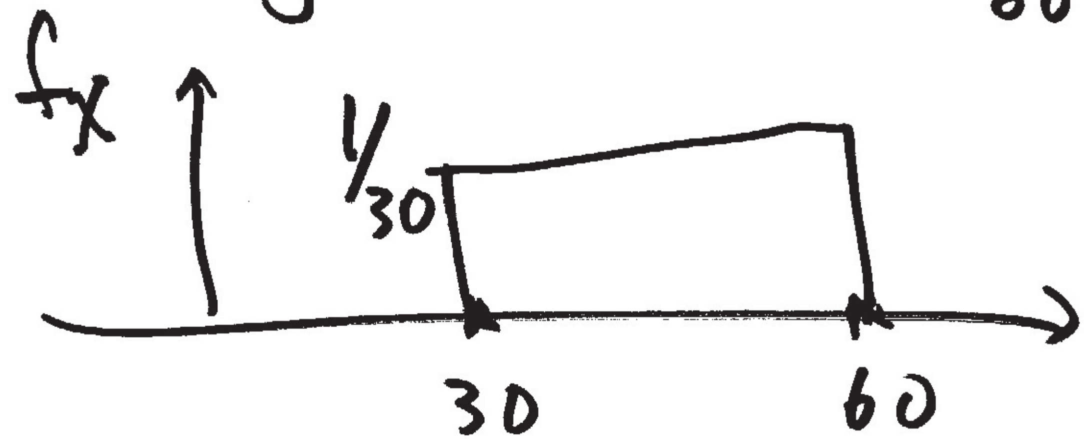
<u>Then</u> pdf of Y in the region $f_Y(y) > 0$ is.

$$f_Y(y) = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|$$

EX

X = speed of vehicle.

uniformly dist. between 30 mph
60 MPH



SF $\rightarrow$ Carmel = 180 Miles.

Pdf of Time it Takes to get to Carmel?

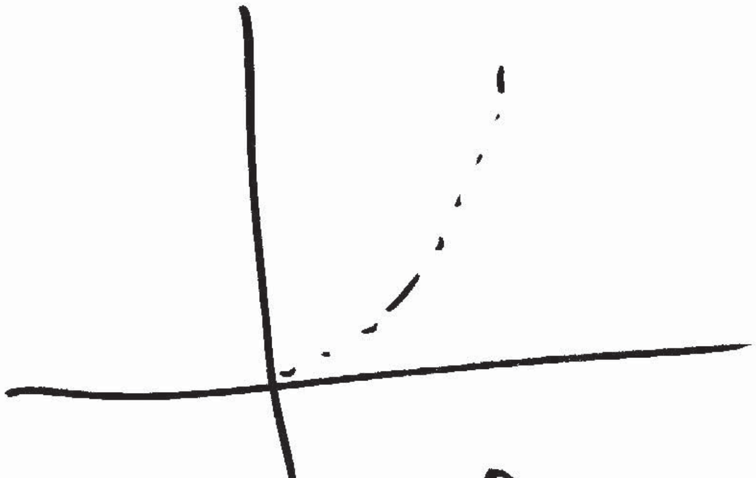
$$y = \frac{180}{X} = g(x) \qquad h(y) = \frac{180}{y}.$$

$$f_Y(y) = \frac{1}{30}\left|\frac{\partial}{\partial y}\left(\frac{180}{y}\right)\right| = \frac{1}{30}\,\frac{180}{y^2} = \begin{cases} \frac{6}{y^2} & 3<y<6 \\ 0 & \text{oth} \end{cases}$$

Ex X uniform R.V. $(0,1]$ includes 1

everything except 0

$g(x) = x^2$



$h(y) = \sqrt{y} \quad \forall\, y \in (0,1]$

$$f_Y(y) = \begin{cases} 1 \cdot \left| \dfrac{dh(y)}{dy} \right| = \dfrac{1}{2\sqrt{y}} & y \in (0,1] \\ 0 & \text{otherwise} \end{cases}$$

2 R.V. g, h.

CDF ① $P_{g\le, h\le}(g_0, h_0)$ $\forall g_0, h_0$

② $$f_{g,h}(g_0, h_0) = \frac{\partial^2 P_{g\le h\le}(g_0, h_0)}{\partial g_0 \, \partial h_0}$$

Ex Y, X uniformly dist. RV. $[0,1]$. independent.

pdf of $Z = \frac{Y}{X} = g(x,y)$.

CDF