Sum of 2 independent Continous R.V.

X, Y cont. R.V. indp.

$W = X + Y$. $\underline{Q}$: find pdf of W?

$$P(W < w \mid X = x) = P(X + Y \leq w \mid X = x)$$

$$= P(x + Y \leq w \mid X = x)$$

indp of X, Y

$$= P(x + Y \leq w)$$

$$P(W < w \mid X = x) = P(Y \leq w - x)$$

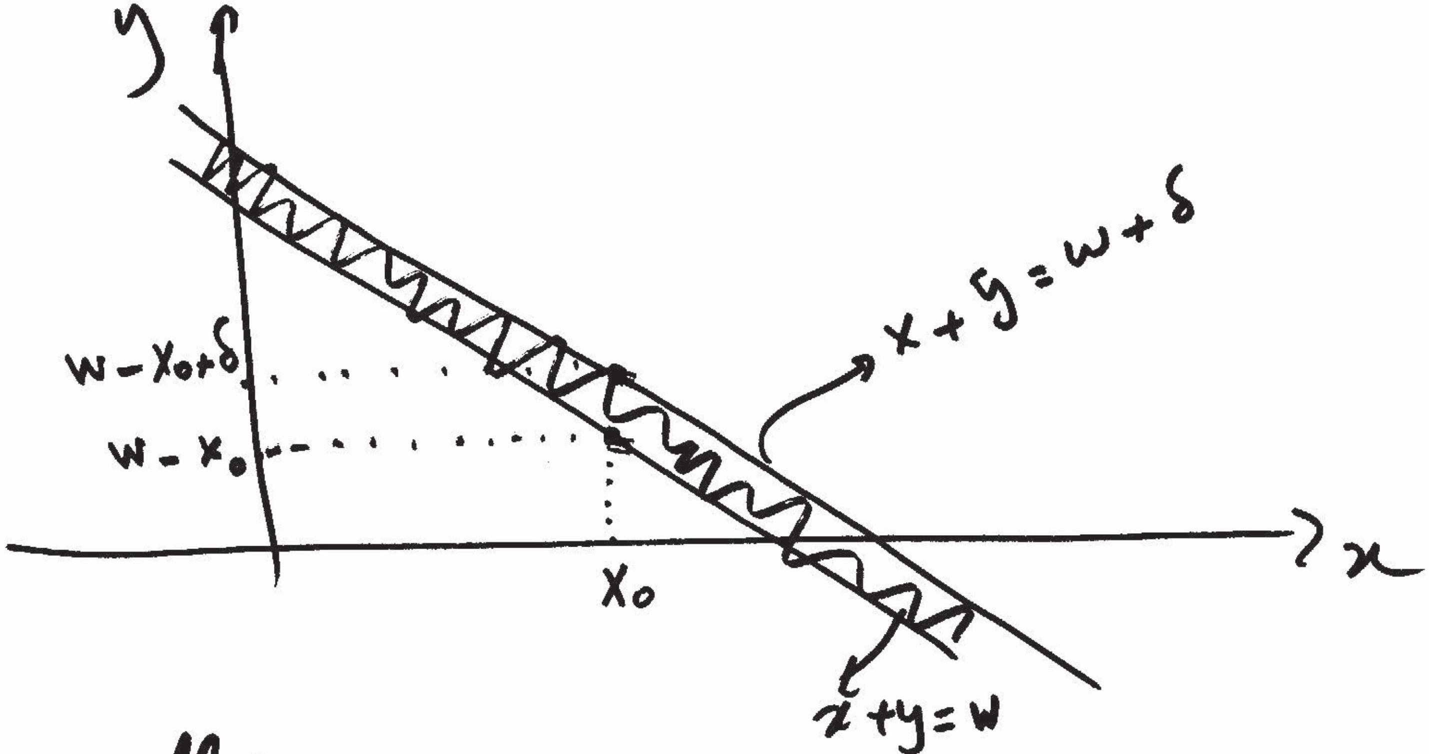
CDF CDF

Differenorintiate both sides

$$\left.\begin{array}{l} f_{W|X}(w|x) = f_Y(w-x) \\ \text{Recall: } f_{W,X}(w,x) = f_{W|X}(w|x)\, f_X(x) \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l} f_{W,X}(w,x) = f_X(x)\, f_Y(w-x) \\ f_W(w) = \int f_{W,X}(w,x)\, dx \end{array}\right\} \Longrightarrow$$

$$\boxed{f_W(w) = \int f_X(x)\, f_Y(w-x)\, dx}$$



δ = small :

$$f_W(w)\,\delta = P\left(w < X + Y < w + \delta \right)$$

$$= \int_{-\infty}^{+\infty} f_X(x_0) \left(\int_{w - x_0}^{w - x_0 + \delta} f_Y(y)\,dy \right) dx_0$$

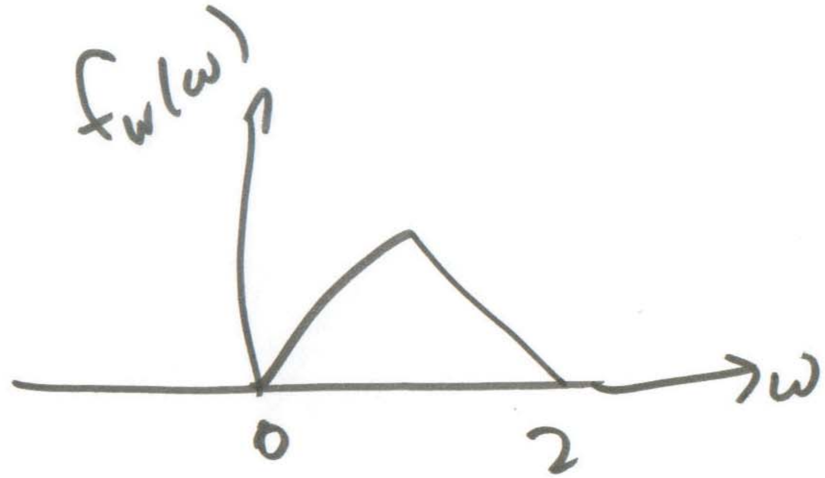
$$f_W(w)\,\delta = \int_{-\infty}^{+\infty} f_X(x_0)\ \delta\ f_Y(w - x_0)\ dx_0$$

$$\Longrightarrow \boxed{f_W(w) = \int_{-\infty}^{+\infty} f_X(x_0)\, f_Y(w - x_0)\, dx_0}$$

Convolution

<u>Ex</u> X, Y indp. uniformly dist $[0,1]$.

$W = X + Y$



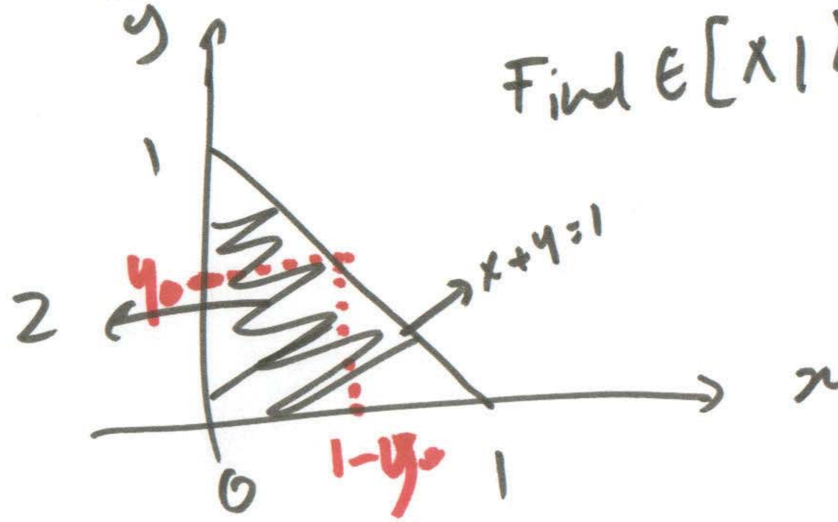
Conditional Expectation + Variance

2 RV. X, Y.

- know $E[X \mid Y=y]$ means.

$$= \begin{cases} \sum_x x \, p_{X|Y}(x|y) & \text{discrete} \\ \int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y)\, dx. & \text{cont.} \end{cases}$$

<u>Ex</u> X, Y joint pdf



Find $E[X \mid Y=y]$

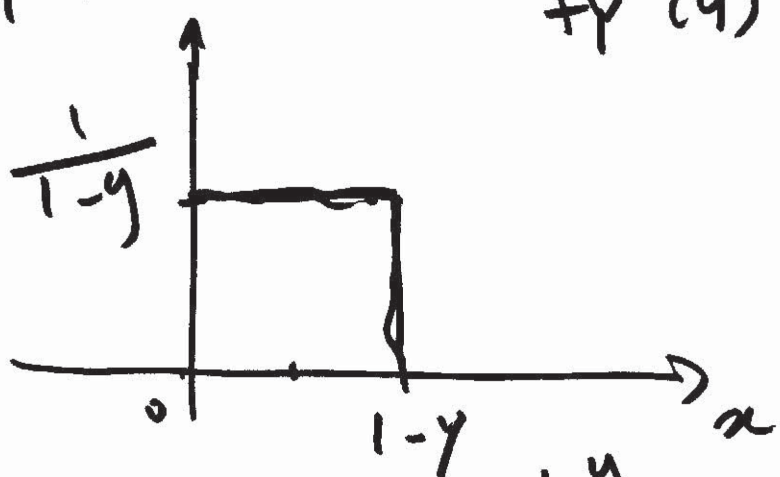
Approach: first compute $f_{X|Y}(x|y) \longrightarrow E[X|Y=y]$.

$$f = \frac{f_{X,Y}}{f_Y}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 2\,dx = 2(1-y) \qquad 0 \le y \le 1$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$0 < x < 1-y$$



$$E[X|Y=y] = \int_0^{1-y} x\, f_{X|Y}(x|y)\,dx$$

$$= \int_0^{1-y} x \frac{1}{1-y}\,dx = \frac{1-y}{2}$$

$$E[X \mid Y = y] = \frac{1-y}{2}$$

$$E[X \mid Y] \leftarrow \text{Define} = \frac{1-Y}{2}$$

A R.V. whose value is $E[X \mid Y=y]$

when value of Y is y.

$E[X \mid Y]$ is a R.V. (fn of Y).

$\Longrightarrow$ has Expectation.

$$E_{Y}\Big[\; E[X \mid Y] \; \Big] =$$

$$= \begin{cases} \displaystyle\sum_y E[X \mid Y=y]\, p_Y(y) & Y \text{ discrete} \\ \displaystyle\int_{-\infty}^{+\infty} f_Y(y)\, E[X \mid Y=y]\, dy & Y \text{ contin} \end{cases}$$

Recall: p. 105: $E(X) = \sum_y p_Y(y)\, E[X \mid Y=y]$

Total Exp Thm

$$\boxed{E\big[\, E[X \mid Y] \,\big] = E[X]}$$

law of iterated expectation

<u>Ex</u> $E[X \mid Y] = \frac{1-Y}{2}$

$$E\left[E[X \mid Y] \right] = E[X]$$

applied law of iterated exp.

$$E\left[\frac{1-Y}{2} \right] = E[X]$$

$$\frac{1}{2} - \frac{E[Y]}{2} = E[X]$$

$$E[X] = E[Y]$$

$$\Longrightarrow E[X] = E[Y] = \frac{1}{3}$$

$$E[\quad] = \int_0^1 f_Y(y)\, E[X \mid Y = y]\, dy$$

$$= \int_0^1 2(1-y)\, \frac{1-y}{2}$$

$$= 1/3$$

Ex stick of length l.

- Break it, uniformly dist. over l
  Keep stick on left after break $\rightarrow$ Y = length. R.V.
- Break left one again, keep left. $\rightarrow$ X = length of stick after 2nd time of breaking

What is $E(X)$?

$$E[X] \overset{\text{law of iter. exp}}{=} E\big[\, E[X \mid Y] \,\big] \quad \Rightarrow$$

$$E[X \mid Y] = \frac{Y}{2}$$

$$E[X] = E\left[\frac{Y}{2}\right] = \frac{1}{2} E[Y] = \frac{1}{2}\,\frac{\ell}{2} = \frac{\ell}{4}$$

Conditional Variance:

$$E\left[\ Var(X \mid Y)\ \right] = ??$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2$$

law of it. exp.

$$= E\left[\ \underbrace{E[X^2 \mid Y]}\ \right]$$

$$- \left(E\left[\ E[X \mid Y]\ \right]\right)^2$$

$$= E\left[\ \underbrace{Var(X \mid Y) + \left(E[X \mid Y]\right)^2}\ \right]$$

$$- \left(E\left[\ E[X \mid Y]\ \right]\right)^2$$

$$= E[\operatorname{Var}(X|Y)] +$$

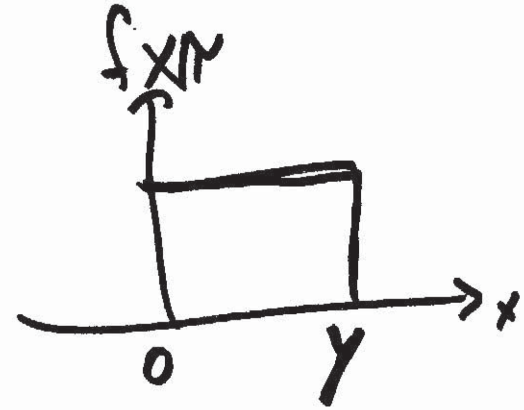
$$\left\{ E\left[\big(\underbrace{E[X|Y]}_{Z}\big)^2 \right] - \left(E\left[\underbrace{E[X|Y]}_{Z} \right] \right)^2 \right. = \operatorname{Var}[E[X|Y]]$$

$$\boxed{\operatorname{Var}(X) = E[\operatorname{Var}(X|Y)] + \operatorname{Var}[E[X|Y]]}$$

law of Total Variance.

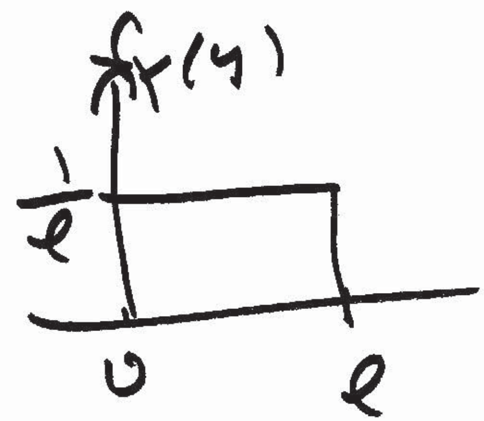
Ex of Breaking stick

What $\mathrm{Var}(X)$?



$$\mathrm{Var}(X \mid Y) = \frac{Y^2}{12}$$

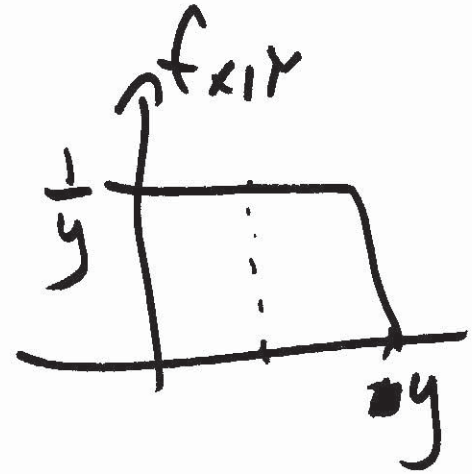
$$E[\mathrm{Var}(X \mid Y)] = E\left[\frac{Y^2}{12}\right] = \frac{1}{12} E[Y^2]$$



$$= \frac{1}{12} \int y^2 f_Y(y)\, dy$$

$$= \frac{1}{12} \int_0^{\ell} y^2 \frac{1}{\ell}\, dy = \frac{\ell^2}{36}$$

$$E[X \mid Y] = \frac{Y}{2}$$



$$\mathrm{Var}\left[E[X \mid Y] \right] = \mathrm{Var}\left[\frac{Y}{2} \right] =$$

$$= \frac{1}{4} \mathrm{Var}[Y].$$

$$= \frac{1}{4} \frac{l^2}{12} = \frac{l^2}{48}$$

$$\mathrm{Var}(X) = \frac{l^2}{36} + \frac{l^2}{48} = \frac{7l^2}{144}$$