Sum of a Random # of indp. Random Var.

I.I.D. R.V. = Independently Identically Distributed. Random Variables.

$X_1 \quad X_2 \quad X_3 \quad \cdots \quad X_n$

Identical distribution = same pdf or pmf.

$\Longrightarrow$ X_i same mean, same variance.

$$Y = X_1 + X_2 + \cdots + X_N$$

N random Variable. same pmf.

$E(Y)$? $\quad Var(Y)$?

$$\rightarrow E(Y / N=n) = E[X_1 + X_2 + \ldots X_N / N=n]$$

$$= E[X_1 + X_2 + \ldots + X_n]$$

$$= n \; E[X_i]$$

$$\Rightarrow E[Y/N] = N\,E[X_i]$$

Apply law of iterated Expectation

$$E[Y] = E[\; E[Y/N]\;] = E[\;N \underbrace{E[X_i]}_{\text{constant.}}\;]$$

$$\boxed{E[Y] = E[X_i)\;E[N]}$$

$$Var(Y \mid N=n) = Var(X_1 + X_2 + \cdots + X_N \mid N=n)$$

$$= Var(X_1 + X_2 + \cdots + X_n)$$

$$\stackrel{\text{indp.}}{=} n \, Var(X_i)$$

Apply law of Total Variance:

$$Var(Y) = E[Var(Y \mid N)] + Var(E[Y \mid N]) \implies$$

$$Var(Y \mid N) = N \, Var(X_i)$$

$$Var(Y) = E[N \, \underline{Var(X_i)}] + Var(N \, \underline{E[X_i]})$$

$$\boxed{Var(Y) = Var(X_i) \, E[N] + (E[X_i])^2 \, Var(N)}$$

Ex - Need to find a book.

- Visit websites till you find it.
- prob (book is on a given web site) = p.
- spend a random amount time on each site

exponential dist with parameter λ.

ith web site. $\rightarrow X_i$

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mean + Var. of Total Time spent.

N = R.V. Total # of websites visited.

X_i = Time on ith web site.

$$Y = X_1 + X_2 + \cdots + X_N$$

$$E(Y) = E(X_i)\,E(N)$$

$E(X_i) = 1/\lambda$

$E(N) = \frac{1}{p}$

$N = \underline{\text{geometric R.V.}}$
p.

$$E(Y) = \frac{1}{\lambda} \cdot \frac{1}{p}$$

$$Var(Y) = E(N)\,Var(X_i) + \left(E(X)\right)^2 \,Var(N)$$

$$= \frac{1}{p} \cdot \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 \frac{1-p}{p^2} = \frac{1}{\lambda^2 p^2}$$

<u>How about Transform of Y.</u>

$$Y = X_1 + \cdots\cdots + X_N$$ ← random

$M_Y(s)$? $= E[e^{sY}]$

$$E[e^{sY} / N = n] = E[e^{sX_1} \cdots\cdots e^{sX_N} \mid N = n]$$

$$= E[e^{sX_1} \cdots\cdots e^{sX_n}]$$

indep.

$$= [M_X(s)]^n$$

$$M_Y(s) = E[e^{sY}] = E[\; E[e^{sY} \mid N] \;]$$

apply law of iterated expect.

$$= E\left[\left(M_X(s)\right)^N \right]$$

$M_Y(s)$

$$= \sum_{n=0}^{\infty} \left(M_X(s)\right)^n p_N(n)$$

$$M_N(s) = E\left[e^{sN}\right] = \sum_{n=0}^{\infty} (e^s)^n \, p_N(n)$$

$$\boxed{M_Y(s) = \Big[\, M_N(s) \,\Big]_{e^s = M_X(s)}}$$

$\underline{Ex}$ find pdf of Y : Total amount of Time spent visiting.

$$M_N(s) = \frac{p e^s}{1-(1-p)e^s} \qquad M_X(s) = \frac{\lambda}{\lambda - s}$$

$$M_Y(s) = \left[\frac{p e^s}{1-(1-p)e^s}\right]_{e^s = \frac{\lambda}{\lambda - s}}$$

$$M_Y(s) = \frac{p\,\frac{\lambda}{\lambda - s}}{1 - (1-p)\frac{\lambda}{\lambda - s}} = \frac{p\lambda}{p\lambda - s}$$

$$f_Y(y) = p\lambda\, e^{-p\lambda y} \qquad y \geq 0$$

exponential.

Conclusion: Add a random # of exp. R.V. $\Longrightarrow$ exp. R.V.

geometric

Ex Consider R.V. k. pmf.

$$P_k(k_0) = \frac{8^{k_0}}{9^{k_0+1}} \qquad k_0 = 0, 1, 2, \ldots\ldots$$

Q ~~Ex~~ mean & Var (k).

Transform. $E[k] = \left[\frac{d}{ds} M_k(s)\right]_{s=0}$

$$\boxed{M_k(s) = \sum_{k_0} \left(e^{s}\right)^{k_0} p_k(k_0)} = \frac{1}{9} \sum_{k_0=0}^{\infty} \left(\frac{8e^s}{9}\right)^{k_0}$$

$$M_k(s) = \frac{1}{9-8e^s}$$

$E[k] = 8$

$$\mathrm{Var}(k) = E(k^2) - E^2(k) = \left[\frac{d^2}{ds^2} M_k(s)\right]_{s=0} - 64$$

$$= 72$$

Q Find prob K is even.

$$\rightarrow P(A) = \sum_{k_0 \text{ even}} p_K(k_0) = \frac{1}{9}\left(1 + \frac{8^2}{9^2} + \frac{8^4}{9^4} + \cdots\right)$$

$$= \frac{1/9}{1 - \frac{64}{81}} = \frac{9}{17}$$

$$\rightarrow P(A) = \sum_{k_0 \text{ even}} p_K(k_0)$$

$$= \frac{1}{2}\Big[\underbrace{\sum_{k_0} p_K(k_0)(1)^{k_0}}_{1} + \underbrace{\sum_{k_0} p_K(k_0)(-1)^{k_0}}_{\left[M_K(s)\right]_{e^s=-1}}\Big]$$

$$P(A) = \frac{1}{2}\left[1 + \frac{1}{9-8(-1)}\right] = \frac{9}{17}$$

Q r = sum of $\underline{n}$ indp. experimentals of r.v. k. What is prob r is even.

$$M_r(s) = \left[M_k(s)\right]^n$$

$$M_r(s) = \left(\frac{1}{9-8e^s}\right)^n$$

$$P(r \text{ even}) = \frac{1}{2}\left[1 + \left[M_r(s)\right]_{e^s=-1}\right]$$

$$= \frac{1}{2}\left\{1 + \left[\left(\frac{1}{9-8e^s}\right)^n\right]_{e^s=-1}\right\}$$

$$P(\text{reven}) = \frac{1}{2} \cdot \left[1 + \left(\frac{1}{17}\right)^n\right]$$

Q K = # of light bulbs at Time T_0.

x = life of each bulb.

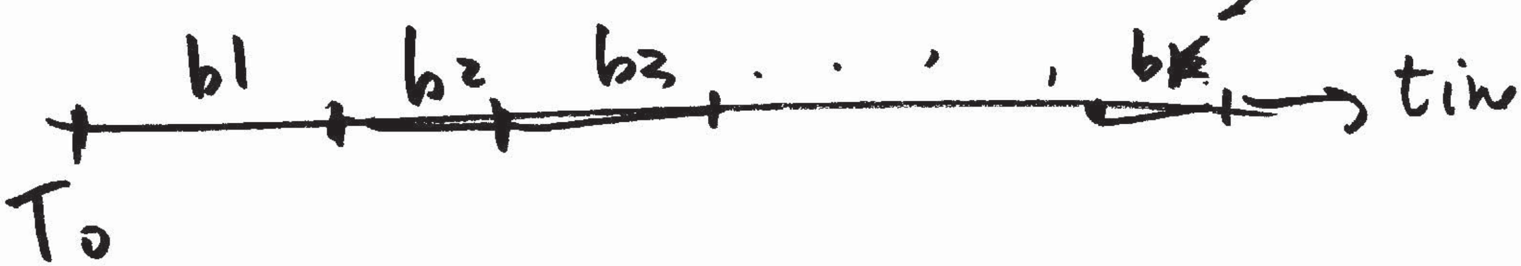
$$f_x(x_0) = \lambda e^{-\lambda x_0} \qquad x_0 \geq 0$$

τ = Time wait until all K bulbs are out.

τ = sum of K indep R.V.

pmf.

Q $E(\tau)$



$$E(\hat{T}) = E(K)\, E(X)$$

$$E(K) = 8 \qquad \sigma_K^2 = 72.$$

$$E(X) = \frac{1}{\lambda} \qquad \sigma_X^2 = \frac{1}{\lambda^2}$$

$$E(\hat{T}) = 8 \cdot \frac{1}{\lambda} = \frac{8}{\lambda}.$$

$$Var(\hat{T}) = E(K)\, \sigma_X^2 + \left(E(X)\right)^2 \sigma_K^2$$

$$= \frac{80}{\lambda^2}$$

$$M_{\hat{T}}(s) = \left[\frac{1}{9-8e^s} \right]_{e^s = \frac{\lambda}{s+\lambda}}$$

$$M_{\hat{T}}(s) = \frac{s+\lambda}{9s+\lambda}$$

$$\Longrightarrow E(\tau) \quad Var(\tau).$$

Q $f_\tau(\tau_0)$

$$Inv\left[\frac{s+\lambda}{as+\lambda}\right] = Inv\left[\frac{1}{a} + \frac{8}{a}\,\frac{\lambda/a}{s+\lambda/a}\right]$$

$$= \frac{1}{a}\delta(\tau_0) + \underbrace{\frac{8}{a}\cdot\frac{\lambda}{a}\, e^{-\lambda\tau_0/a}}_{\tau_0>0}\, \underbrace{u(\tau_0)}_{\text{step fn.}}$$