Transforms

R.V. X.

Transform or moment generating fn is defined as.

$$M_X(\underline{s}) = E[e^{sX}]$$

R.V.

<u>Discrete case</u> $\quad M_X(s) = \sum_x e^{sx} p_X(x)$

<u>Continous Case</u>: $\quad M_X(s) = \int_{-\infty}^{+\infty} e^{sx} f_X(x) \, dx$

Ex

$$f_x(x_0) = \begin{cases} \lambda e^{-\lambda x_0} & x_0 \geq 0 \\ 0 & x_0 < 0 \end{cases}$$

$$M_x(s) = \int_{-\infty}^{+\infty} e^{-s x_0} f_x(x_0) = \int_0^{\infty} \lambda e^{+s x_0} e^{-\lambda x_0} dx_0$$

$$= \frac{\lambda}{\lambda - s}$$

Ex Poisson R.V. Discrete.

$$P_X(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, \ldots$$

$$M(s) = \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!} \qquad \text{let } a = e^s \lambda$$

$$M(s) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{a^x}{x!} = e^{-\lambda} e^{a} = e^{a-\lambda}$$

$$= e^{\lambda(e^s - 1)}$$

Transform of a linear fn of a R.V.

$y = ax + b$

Known: Transform of X.

Q what is xform of y.

$$M_Y(s) = E\left[e^{s(ax+b)}\right] = e^{sb} \, E\left[e^{sax}\right]$$

$$M_Y(s) = e^{sb} M_X(as)$$

Ex X cont. R.V. : exponential $\lambda = 1$

$$M_X(s) = \frac{1}{1-s} \qquad Y = 2X + 3$$

$$M_Y(s) = e^{3s} \frac{1}{1-2s}$$

Ex standard normal R.V.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \longrightarrow M_Y(s) = e^{s^2/2}$$

~~Ex~~ Normal R.V. $X = \sigma Y + \mu$

$\rightarrow \text{Normal} \sim (\underset{\text{mean}}{\mu}, \underset{\text{variance}}{\sigma^2})$

$$M_X(s) = e^{s\mu} M_Y(s\sigma) = e^{s\mu} e^{\frac{\sigma^2 s^2}{2}}$$

$$M_X(s) = e^{\frac{\sigma^2 s^2}{2} + s\mu}$$

Computing moments using Xforms

$$\frac{d}{ds}\left(M(s)\right) = \frac{d}{ds}\left(\int e^{sx} f_X(x)\,dx\right)$$

$$= \int \frac{d}{ds}\left(f_X(x)\, e^{sx}\right) dx$$

$$\frac{d}{ds}\left[M(s)\right] = \int f_X(x)\; x\, e^{sx}\; dx \qquad \forall s$$

let $s=0$ $\quad \left[\frac{d}{ds} M(s)\right]_{s=0} = \int x\, f_X(x)\,dx = E[X]$

$$\boxed{E(X) = \left[\frac{d}{ds} M(s)\right]_{s=0}}$$

Can show: $$\left[\frac{d^n M(s)}{ds^n}\right]_{s=0} = \int_{-\infty}^{+\infty} x^n f_X(x)\,dx$$

$$= E[X^n]$$

$$Var(X) = E[X^2] - (E(X))^2$$

EX $$P_X(x) = \begin{cases} 1/2 & x=2 \\ 1/6 & x=3 \\ 1/3 & x=5 \end{cases}$$

$$M(s) = \frac{1}{2}e^{2s} + \frac{1}{6}e^{3s} + \frac{1}{3}e^{5s}$$

$$E(X) = \left[\frac{d}{ds} M(s)\right]_{s=0} = \left[\frac{1}{2} 2e^{2s} + \frac{1}{6} \cdot 3e^{3s} + \frac{1}{3} 5e^{5s}\right]_{s=0}$$

$$= \frac{1}{2} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{3} \cdot 5$$

$$= \frac{19}{6}$$

$$E[X^2] = \left[\frac{d^2}{ds^2} M(s)\right]_{s=0} = \left[\frac{1}{2} 4e^{2s} + \frac{1}{6} 9e^{3s} + \frac{1}{3} 25e^{5s}\right]_{s=0}$$

$$= \frac{71}{6}$$

Ex exp. R.V.

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

$$M(s) = \frac{\lambda}{\lambda - s}$$

$$\frac{d}{ds}[M(s)] = \frac{\lambda}{(\lambda - s)^2}$$

$$\frac{d^2}{ds^2}[M(s)] = \frac{2\lambda}{(\lambda - s)^3}$$

$$E(X) = \frac{1}{\lambda} \qquad E[X^2] = \frac{2\lambda}{(\lambda - s)^3} = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Inversion

- pattern matching.. / look up at table.

Ex

$$M(s) = \sum_x e^{sx} P_X(x)$$

$$M(s) = \frac{1}{4} e^{-s} + \frac{1}{2} + \frac{1}{8} e^{4s} + \frac{1}{8} e^{5s}$$

$$= \dots + e^{-s} P_X(-1) + e^{s0} P_X(0) + e^{s1} P_X(1) + e^{2s} P_X(2) + \dots e^{4s} P_X(4)$$

$$\Rightarrow P_X(x) = \begin{cases} \frac{1}{2} & X = 0 \\ 1/8 & X = 4 \\ 1/8 & X = 5 \\ 1/4 & X = -1 \end{cases}$$

EX $\quad f_X(x) = \frac{2}{3}\cdot 6 e^{-6x} + \frac{1}{3} 4 \cdot e^{-4x}.$

$$M(s) = \frac{2}{3}\int_0^\infty e^{sx}\, 6 e^{-6x}\, dx + \int_0^\infty \frac{1}{3} e^{sx} e^{-4x}\, dx$$

$$M(s) = \frac{2}{3}\,\frac{6}{6-s} + \frac{1}{3}\,\frac{4}{4-s}$$

$$\boxed{\text{Inverse}\left\{\frac{1}{\alpha - s}\right\} = e^{-\alpha x}}$$ ← Remember

$\underline{EX}$ $M(s) = \dfrac{pe^s}{1-(1-p)e^s}$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots\cdots$$

$$M(s) = \underline{p}e^s\left[1 + \underline{(1-p)}e^s + \underline{(1-p)^2}\,e^{2s} + (1-p)^3 e^{3s} + \cdots\right]$$

$$M(s) = \sum_x e^{sx}\, P_X(x)$$

$P(X=1) = p$ $\leftarrow$ equating coeff of powers of e^s

$P(X=2) = p(1-p)$ equate coeff of power of e^{2s}

$P(X=3) = p(1-p)^2$ equate coeff of e^{3s}

$$P(X=k) = p(1-p)^{k-1}$$

Sum of indp. R.V.

X, Y indp. $\qquad W = X+Y$.

$$M_W(s) = E[e^{sW}] = E[e^{sX+Ys}]$$

$$= E[e^{sX} e^{sY}]$$

Assume X, Y indp.

$$= E[e^{sX}]\, E[e^{sY}]$$

$$= M_X(s)\, M_Y(s)$$

$$X+Y \longleftrightarrow M_X(s)\, M_Y(s)$$

Ex Bernoulli R.V. with parameter p.

$$M_{X_i}(s) = (1-p)e^{0s} + pe^{1s} = 1-p+pe^{s}$$

$$Y = X_1 + X_2 + \cdots + X_n$$

Binomial

$$M_Y(s) = (1-p+pe^{s})^n$$

$X_1, \ldots, X_n$.

$$M_{X_1, X_2, \ldots, X_n}(s_1, s_2, \ldots, s_n) \triangleq E\left[e^{s_1 x_1 + s_2 x_2 + \cdots + s_n x_n}\right]$$

Sum of Indp. R.V.

$$\boxed{W = X + Y} \quad \text{indp.}$$

Discrete Case:

$$P_W(w) = P(X + Y = w)$$

$$= \sum_{\{(x,y) \mid x+y=w\}} P(X = x, Y = y)$$

$$= \sum_x P(X = x, Y = w - x)$$

indp.

$$P_W(w) = \sum_x P_X(x) \, P_Y(w - x)$$

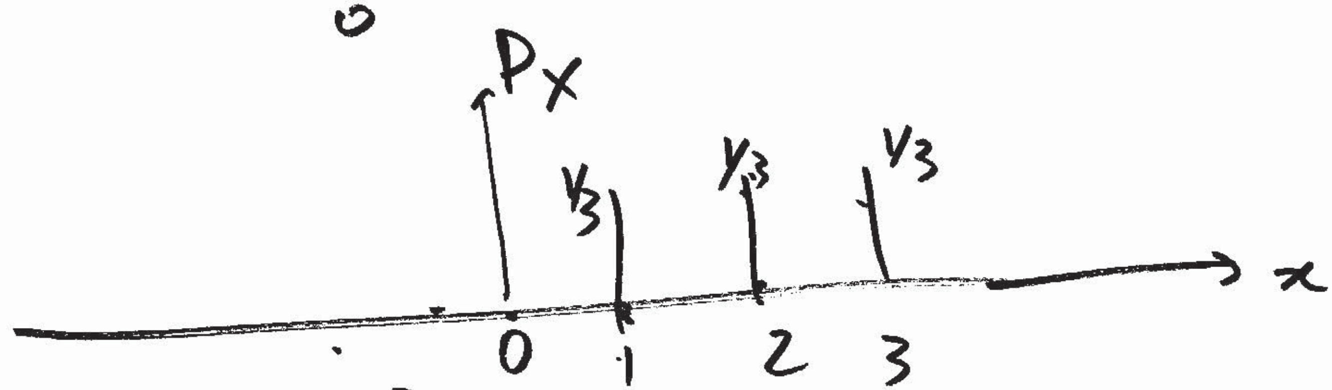
$$\boxed{P_W(w) = P_X * P_Y}$$

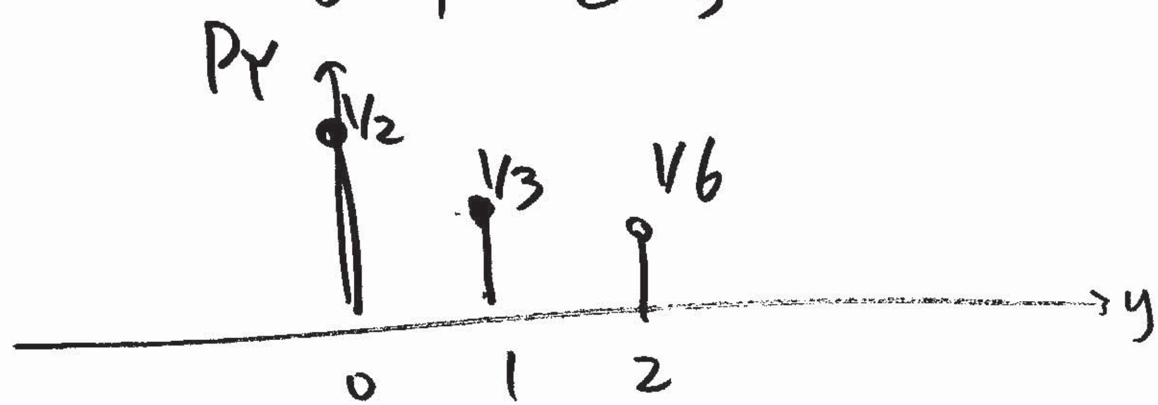
convolution

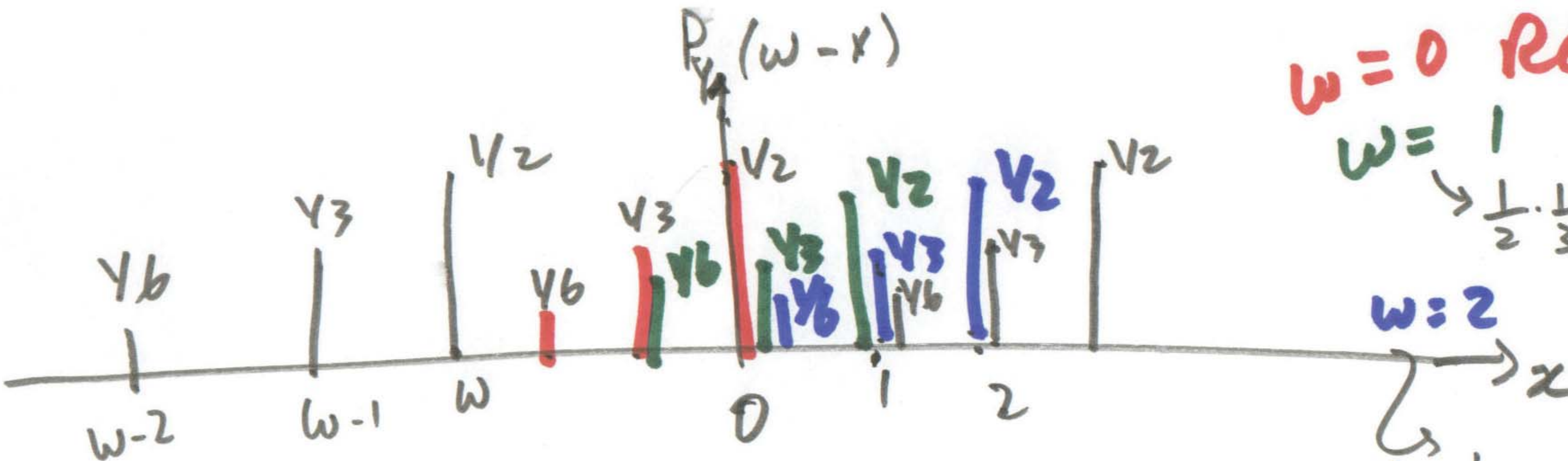
Ex

$$P_X(x) = \begin{cases} 1/3 & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} 1/2 & y = 0 \\ 1/3 & y = 1 \\ 1/6 & y = 2 \\ 0 & \text{otherwise.} \end{cases}$$







$w=0$ Red

$w=1$

$\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$

$w=2$

$\frac{1}{2}\cdot\frac{1}{3}+$

$\frac{1}{3}\cdot\frac{1}{3}=5/18$

$$P_W(w)=\begin{cases} 0 & w\le 0 \\ \frac{1}{6} & w=1 \\ 5/18 & w=2 \\ 1/3 & w=3 \\ 1/6 & w=4 \\ 1/18 & w=5 \end{cases}$$

$w=3$

$\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{3}$

$=$

$$P(W=3) = p(X=3, Y=0) + p(X=2, Y=1) + P(X=1, Y=2)$$

Aliter

indp

$$= p(X=3)\,p(Y=0) + p(X=2)\,p(Y=1) + p(X=1)\,p(Y=2)$$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{6}.$$