Bernoulli Trial

$$P_x(x_0) = \begin{cases} 1-P & x_0 = 0 \to \text{failure.} \\ P & x_0 = 1 \to \text{success.} \\ 0 & \text{otherwise.} \end{cases}$$

$$M_x(s) = 1 - P + e^s P = \sum_{x_0} p_x(x_0)\, e^{s x_0}$$

$$E(x) = P \qquad \sigma_x^2 = p(1-p)$$

Bernoulli Process

Def: Series of indpendent Bernoulli Trials each with the same prob of success.

- n indp. Bern. Trials.

k = R.V. # of successes in n Trials.

K = sum of n indp. Bern. R.V.

what is $P_K(k_0)$?

method 1 : $M_K(s) = \left[M_x(s)\right]^n = (1-P+e^s P)^n$

Recall : $(a+b)^n = \sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l}$ $\Longrightarrow$

$M_K(s) \triangleq P_K(0) + e^s P_K(1) + P_K(2)e^{2s} + \dots$

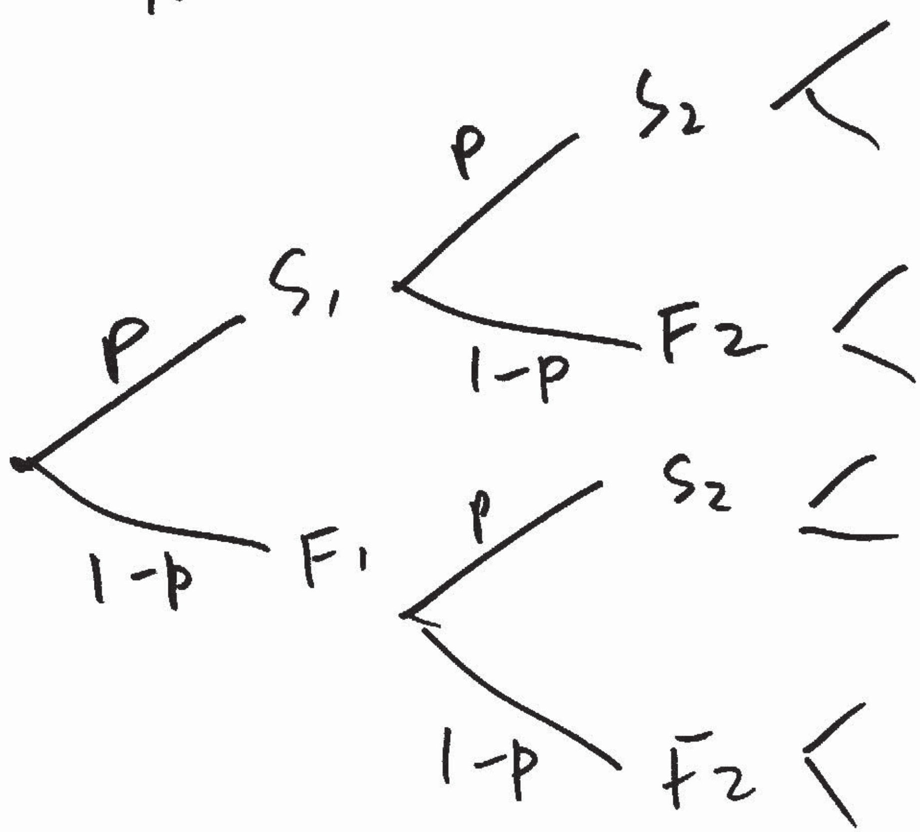
$\Longrightarrow$ equate coeff of power of e^s

$$P_K(k_0) = \binom{n}{k_0} P^{k_0} (1-P)^{n-k_0}$$

$$\binom{n}{K_0} \triangleq \frac{n!}{(n-K_0)!\,K_0!}$$

<u>method 2</u> : look into sequential sample space.

in an experiment consisting of n indep. Trials.



$$\begin{pmatrix} S_n \\ F_n \end{pmatrix} = \left\{ \begin{matrix} \text{success} \\ \text{failure} \end{matrix} \right\} \text{ on}$$

The nth Trial.

outcome of exactly k_0 success at a leaf of the tree $= p^{k_0}(1-p)^{n-k_0}$

- But, there are $\binom{n}{k_0}$ such leaves.

$$\Rightarrow P_K(k_0) = \binom{n}{k_0} p^{k_0}(1-p)^{n-k_0} \qquad k_0 = 0, 1, 2, \dots$$

Binomail R.V.

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how to compute mean + Variance.

3 methods
1. directly use def.
2. Trafom.
3. K = sum of n indp. R.V. X_i $i = 1, \dots, n$

Bern. Trial R.V.
1 = success p.
0 = failure $1-p$

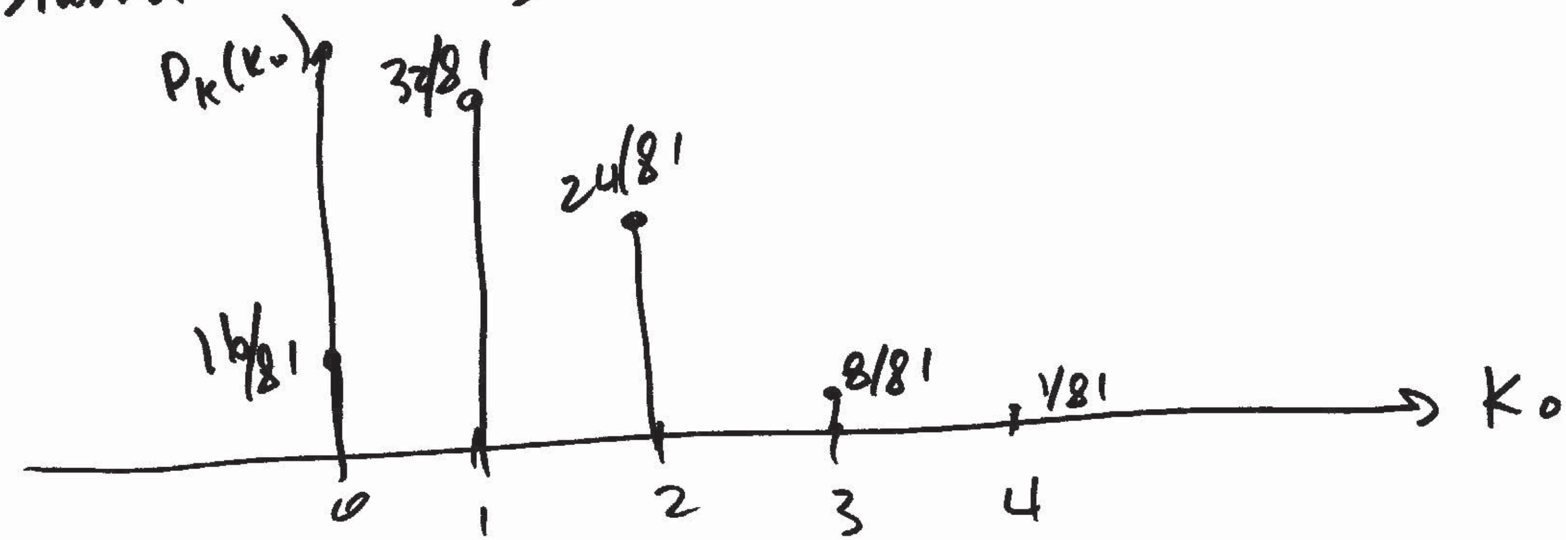
$X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5 \quad \dots \quad X_n$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{n \text{ of them}}$

$$E(K) = n\,E(X) = np.$$

$$Var(K) = n\,Var(X) = n\,p(1-p)$$

<u>Ex</u> plot Binomial. $p = 1/3$ $n = 4$



$$P_K(k_0) = \binom{4}{k_0}\left(\frac{1}{3}\right)^{k_0}\left(\frac{2}{3}\right)^{4-k_0}$$

Inter-arrival Times for Bern. Process

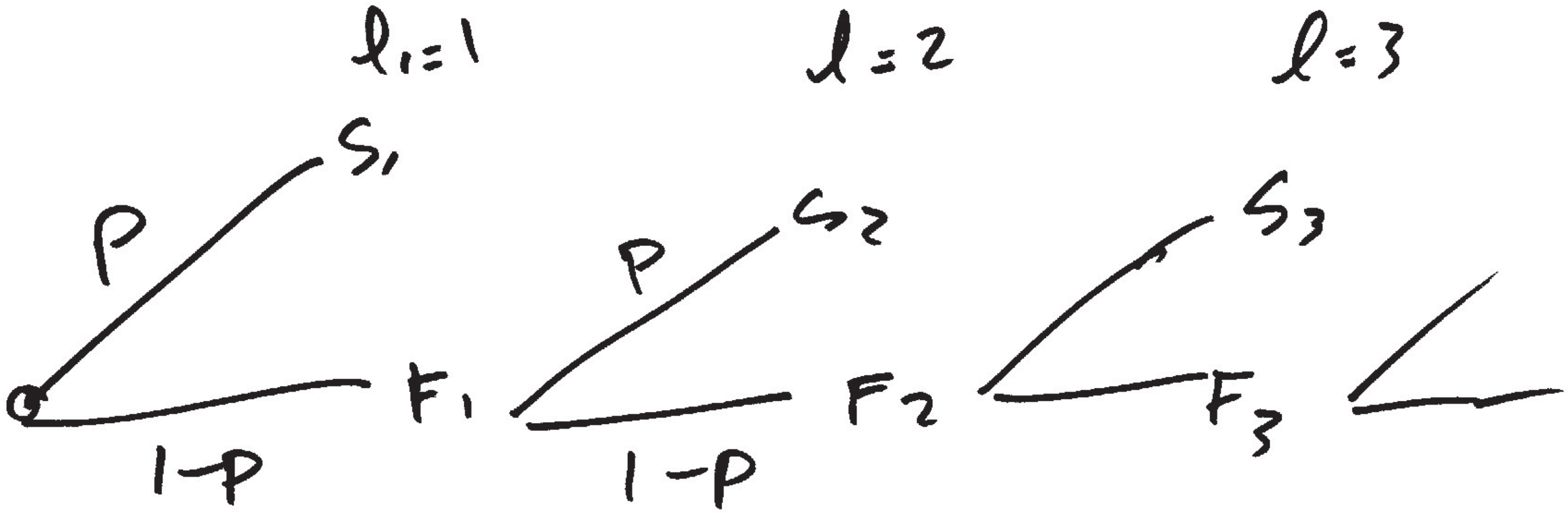
Def success in Bernoulli process is defined as an **Arrival**

l_1 = discrete R.V. # of Bernoulli trials up to and including the first success.

$\triangleq$ first order interarrival time.

takes on discrete values 1, 2, 3, ...

what is pmf for l_1 ?



$$P_{l_1}(l) = (1-P)^{l-1}\, P \qquad l = 1, 2, \dots$$

geometric PMF.

Tranfon:

$$M_{l_1}(s) = \sum_{l=0}^{\infty} P_{l_1}(l)\, e^{sl} = \sum_{l=1}^{\infty} P(1-P)^{l-1} e^{sl} = \frac{e^{s} P}{1 - e^{s}(1-P)}$$

$$\mathbb{E}(l_1) = \left[\frac{d}{ds} M_{l_1}(s) \right]_{s=0} = \frac{1}{p}$$

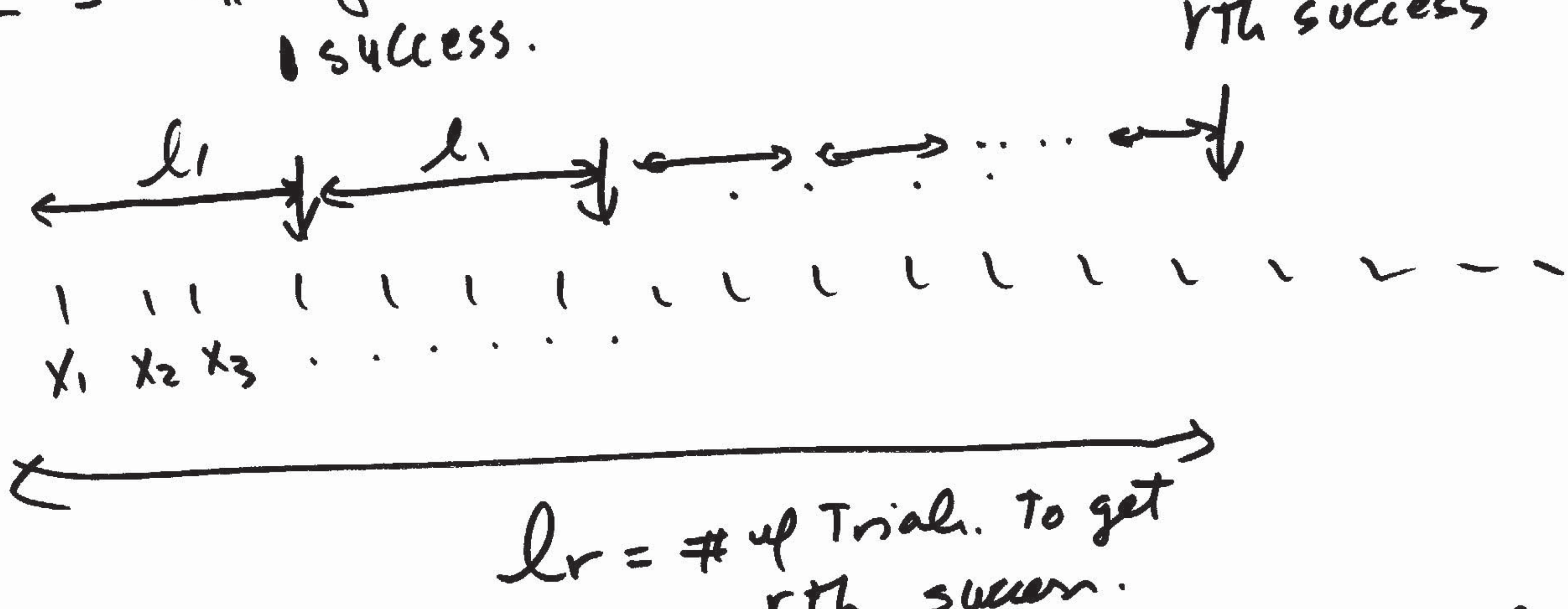
$$\sigma_l^2 = \frac{1-p}{p^2}$$

<u>Fresh start</u>: For any given time n, seq of R.V. $X_{n+1}, X_{n+2}, \ldots \ldots$ (The future process) is also a Bern. process and is indp. from $X_1 \ldots \ldots X_n$ (past process)

<u>Memoryless</u>: Given no success in first m trials what is the conditional pmf for first success?

rth order InterArrival Time?

l_r = # of trials up to and including rth



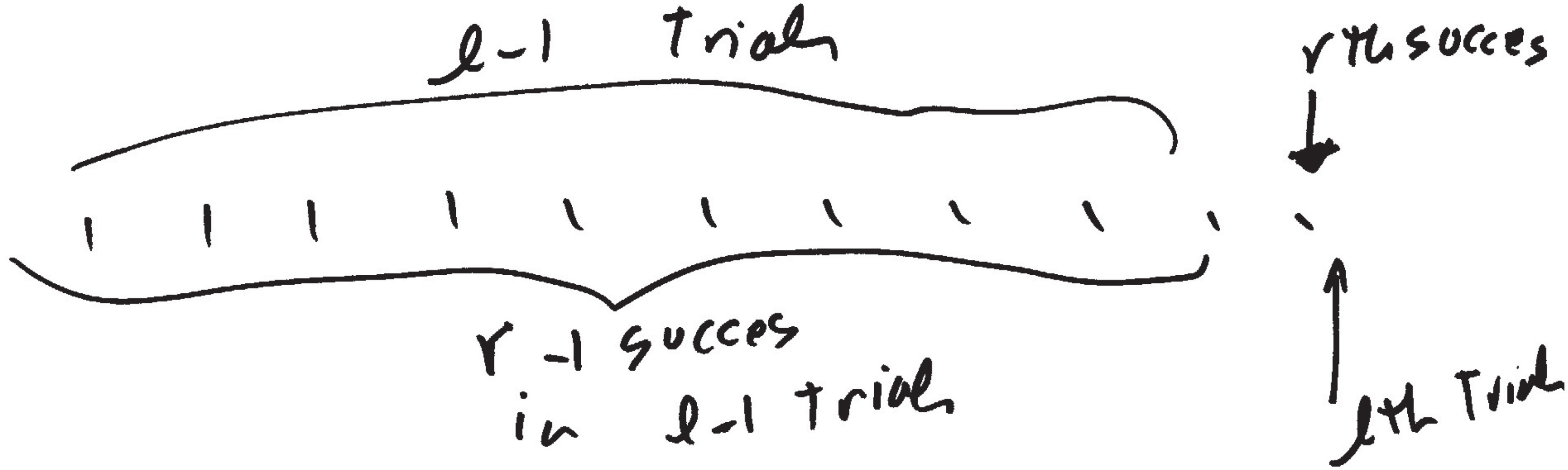
l_r = sum of r indp. experimental values of R.V. l_1

$$M_{l_r}(s) = \left[M_{l_1}(s) \right]^r = \left[\frac{e^s p}{1 - e^s (1-p)} \right]^r$$

Need to compute $P_{l_r}(l)$ = pmf for l_r.

= prob rth success is at lth Trial.

$P_{l_r}(l) =$ (Prob of having exactly $r-1$ successes in The first $l-1$ Trials.) $\times$ (Given exactly $r-1$ successes in The previous $l-1$ trials, Conditional prob of having rth success on The lth Trial.)



=

$$P_{l_r}(l) = \binom{l-1}{r-1} p^{r-1} (1-p)^{l-1-(r-1)} \qquad \downarrow \; p$$

$$P_{l_r}(l) = \binom{l-1}{r-1} p^{r} (1-p)^{l-r}$$

$$l = r,\ r+1,\ r+2, \ldots$$

$$\Longrightarrow E(l_r) = r\,E(l_1) = \frac{r}{p}$$

$$\sigma^2(l_r) = r\,\sigma^2_{l} = \frac{r(1-p)}{p^2}$$

Ex Fred gives out samples of dog food door to door.

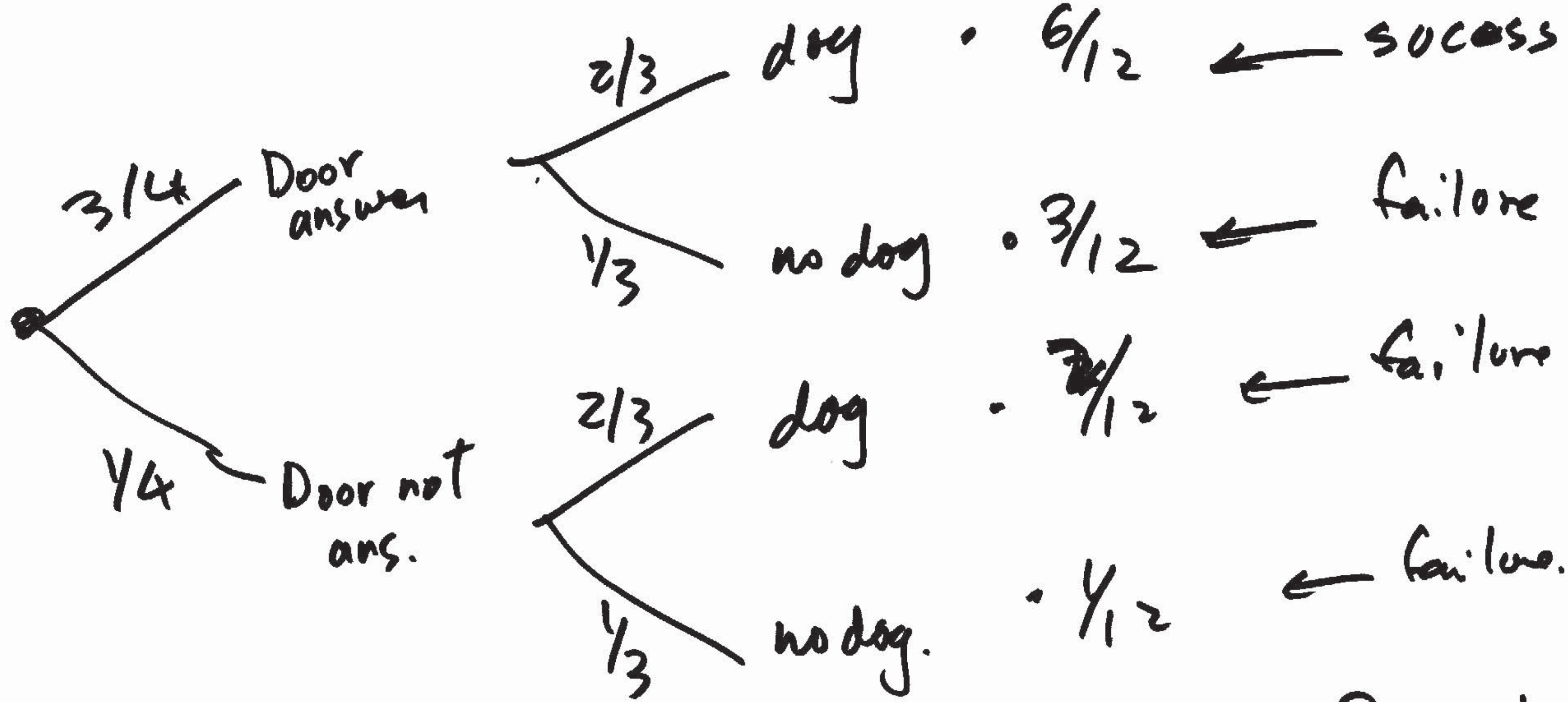
- leave a sample ~~on~~ if.
  - door is answered **AND**
  - There is a dog

P(door answer) = 3/4 P(dog in residence) = 2/3

door answer + having dog are indep.

Q Prob that gives away his 1st sample in his 3rd call?

$$(1-P)^2 P = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1/8 .$$

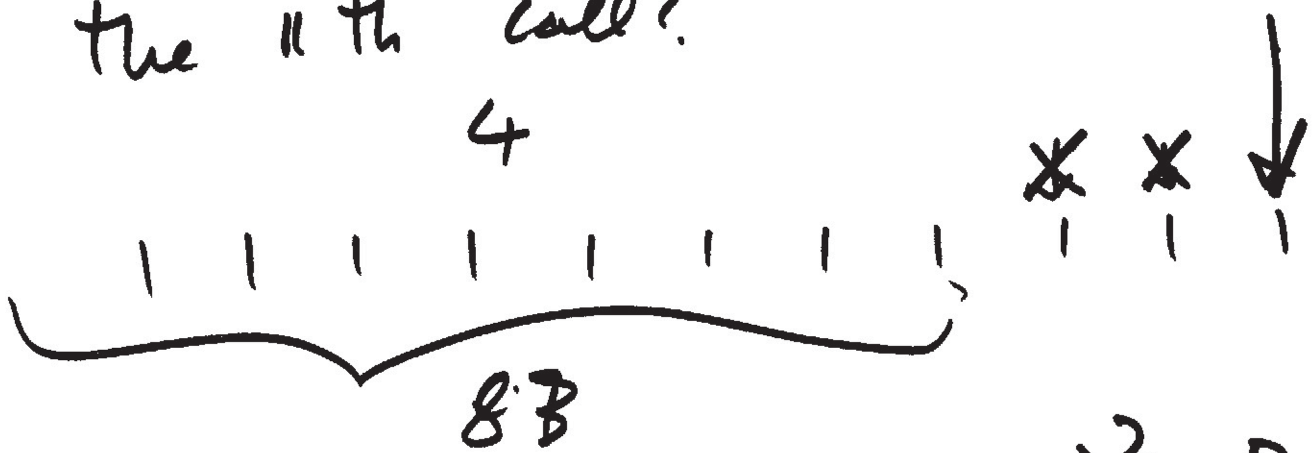


Bernouli process = Prob success $\triangleq P = \frac{1}{2}$

prob failure $= 1-P$

success = handing in a can of dog food.

Q. Given that he has given away exactly 4 samples on his 1st eight calls, what is the conditional prob that he will give away his 5th sample on the 11th call?



$$(1-P)^2 \, P$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Q Pr(gives away his 2nd sample on 5th call) =

$$P_{\ell_2}(\ell) = \binom{\ell-1}{2-1} p^2 (1-p)^{\ell-2}$$

$$P_{\ell_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2}$$

$p = 1/2$

$$= \frac{1}{8}$$

① Given that he did not give away his 2nd sample on his 2nd call, what is the conditional prob. that he will leave his 2nd sample on 5th call?

$$P_{l_2 | l_2 > 2}\left(l_2 = 5 \;/\; l_2 > 2 \right) =$$

$$= \frac{P(l_2 = 5 \text{ and } l_2 > 2)}{P(l_2 > 2)}$$

$$= \frac{P(l_2 = 5)}{1 - P_{l_2}(2)}$$

$$= \frac{\binom{5-1}{2-1} p^2 (1-p)^{5-2}}{1 - \binom{2-1}{2-1} p^2 (1-p)^0} = 1/6$$

__Q__ __Det__ needs a new supply immediately after giving away the last can.

starts 2 can.

Prob __at least__ 5 calls before he needs new supply.

__A__ $P(l_2 \geqslant 5) = 1 - P(l_2 \leq 4)$

$$= 1 - \sum_{\ell=2}^{4} \binom{\ell - 1}{2 - 1} p^2 (1-p)^{\ell-2}$$

$$= 5/16$$