Bernoulli Process

splitting

original → prob success $= p$.

keep q

not keep. $1-q$.

when There is arrival in original. Keep it with prob q.

↓

new Bernoulli process with prob. pq.

when There is arrival in The original process. discard it prob $1-q$.

↓

new Bernoulli process with $p(1-q)$.

Splitting a Bernoulli: results in 2 new Bernoullis.

Merging Two Bernoullis results in a new Bernoulli.

- Start with 2 indp. Berni process.
  p. q.

merge = arrival iff. There is an arrival at least in one of them

$\Rightarrow$ Bernolli with prob
$1-(1-p)(1-q) = p+q-qp$.

Poisson Approx To Binomial

Binomial: $P_B(k) = \frac{n!}{(n-k)!\,(k!)} p^k (1-p)^{n-k}$

Poisson: $P_P(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $\quad k = 0, \ldots$

$$\boxed{n \longrightarrow \infty \qquad p = \lambda/n} \Longrightarrow \text{Then}$$

Poisson nicely approximates Binomial.

Ex Giants win with prob 0.99%. – Play 100 games:
one game.

Q What is the prob. win 100 games?

Bernoulli

$$P_x(0) = (1-0.01)^{100} = 0.99^{100} = 0.366$$

poission: $\lambda = np = 1 \rightsquigarrow e^{-1}\frac{1}{0!} = 0.368$

Q prob lose 2 games?

$$\frac{100!}{98!\,2!}(0.01)^2(1-0.01)^{98} = 0.185$$

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poisson: $e^{-1}\frac{1}{2!} = 0.184$

Poisson Process

fish.

Poisson: arrivals at points on a Continous line

Show: Poission is Bernouli process. one Trial per Δt, prob(success on a Trial) = $\lambda \Delta t$
limit $\Delta t \to 0$:

Notation $P(k;t)$ = prob of k arrivals during any interval of duration t

PMF for R.V. k.

$$\sum_{k=0}^{\infty} P(k;t) = 1 \qquad \text{for fixed } t.$$

Define ① Any events on non overlapping. Time intervals are mutually indp.

$\Longrightarrow$ no memory.

② for small Δt we assume

$$P(k,\Delta t) = \begin{cases} 1-\lambda\Delta t & k=0 \\ \lambda\Delta t & k=1 \\ 0 & \text{otherwise.} \end{cases}$$

$$P(k;\,t+\Delta t) =$$

$$= P(k;t)\,P(0;\Delta t) + P(k-1;t)\,P(1;\Delta t)$$

$$P(k;\,t+\Delta t) = P(k;t)(1-\lambda\Delta t) + P(k-1;t)\,\lambda\Delta t$$

$$\underbrace{\frac{P(k;t+\Delta t)}{\Delta t} - \frac{P(k;t)}{\Delta t}} + \lambda P(k;t) = \lambda P(k-1;t)$$

$\lim \Delta t \to 0$

$$\frac{d}{dt}P(k;t) + \lambda P(k;t) = \lambda P(k-1;t)$$

Initial Condition:

$$P(K, 0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Soln :

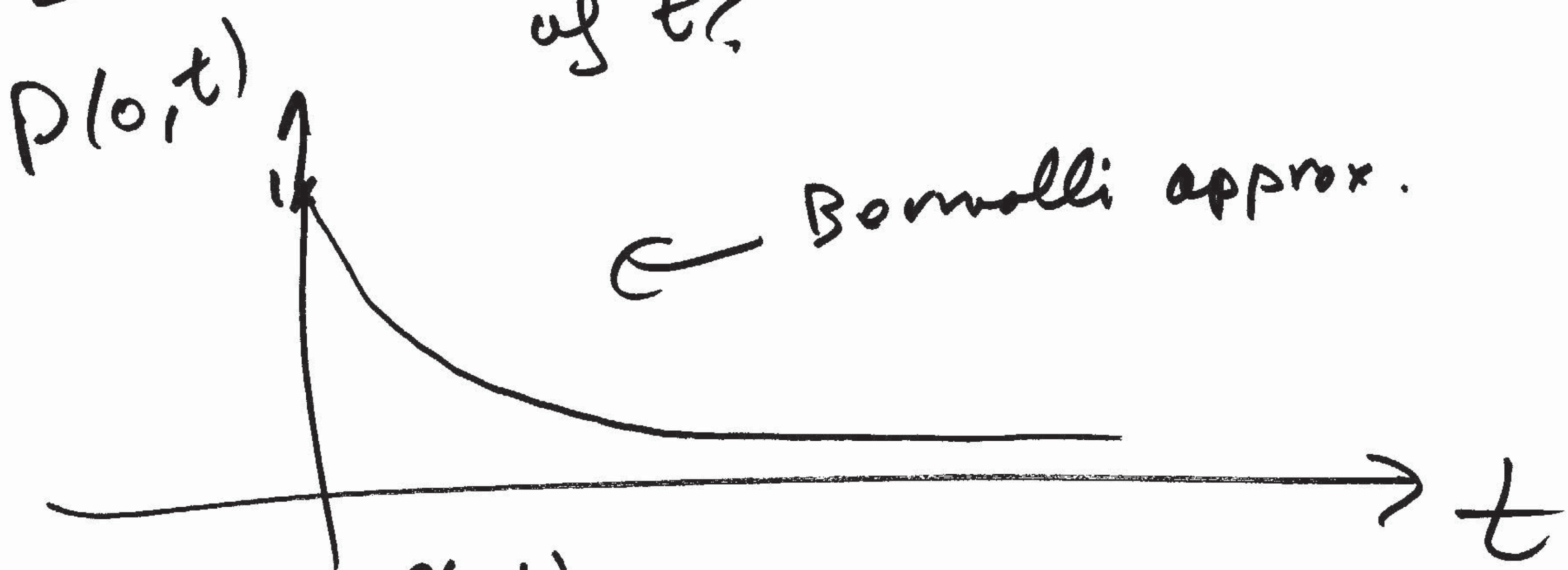
$$\boxed{P(k;t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}} \qquad \begin{array}{l} t \geq 0 \\ k = 0, 1, 2, \ldots \end{array}$$

verify this by substitution

$\mu = \lambda t$ arrival rate.

$$\boxed{P_k(k_0) = \frac{\mu^{k_0} e^{-\mu}}{k_0!}}$$

Q1 how should $P(0,t)$ behave as a fn of t?

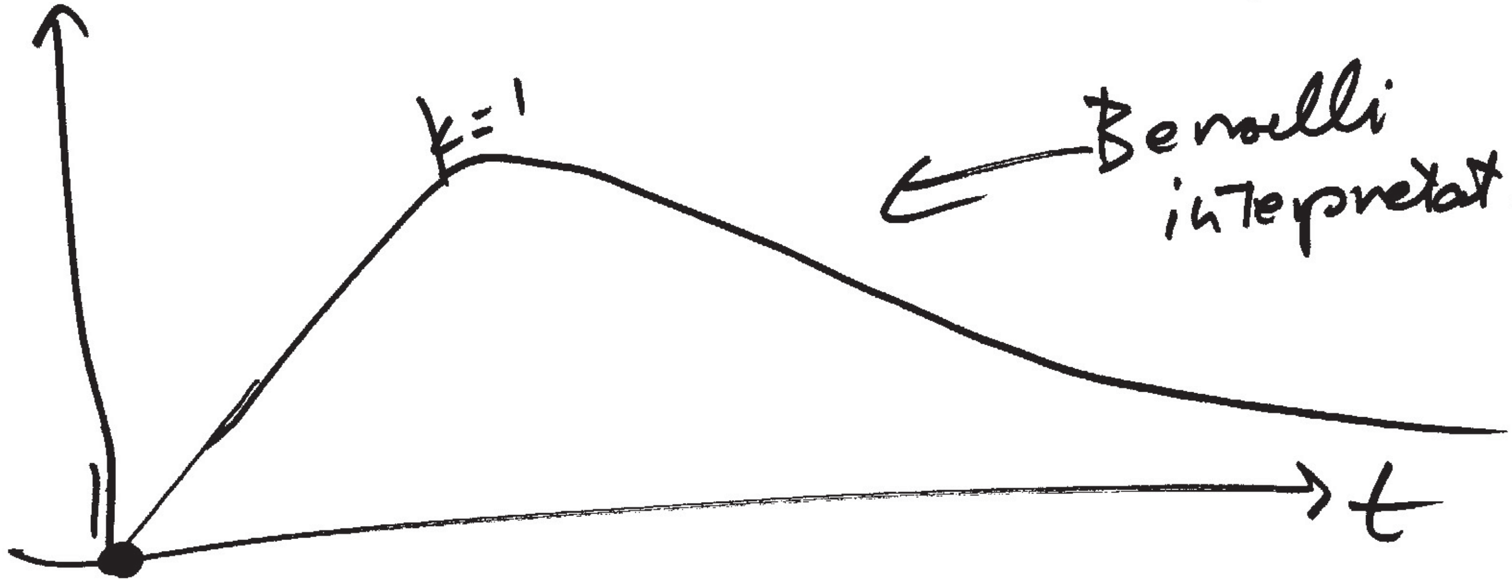


Eqn
let k=0
$e^{-\lambda t}$

Q2

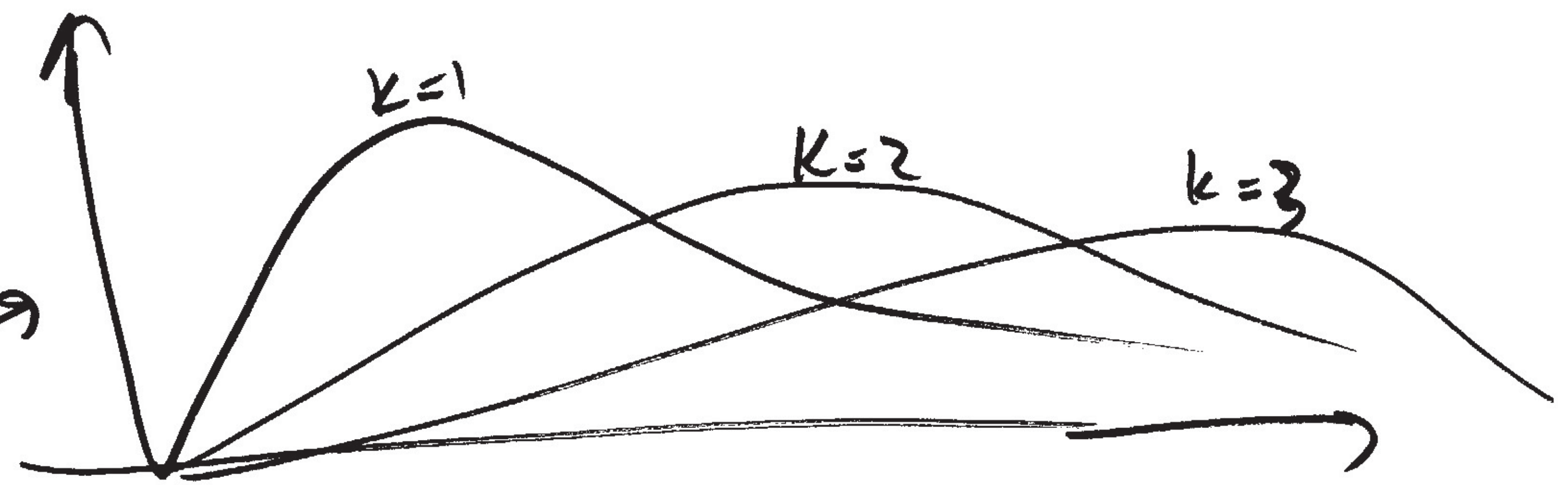
$P(k,t)$ $k \neq 0$ as a fn of t

$k=1$



$k>1$

formula



Tranform. $E(k) = \cdots\cdots = \mu = \lambda t$

$$\sigma_k^2 = \mu$$

Aliter $E(k)$:

for Δt : Expected # of arrivals :

$$0 \cdot (1 - \lambda \Delta t) + 1 \cdot \lambda \Delta t = \lambda \Delta t .$$

Time interval t has $\frac{t}{\Delta t}$ intervals of Δt long.

$$E(k) = \lambda \Delta t \cdot \frac{t}{\Delta t} = \lambda t .$$

Ex $\lambda = 0.2$ message / hr.

arrival rate

What is Prob of finding 0 messages in 1 hour.

$$P(K,t) = P(0,1) = \frac{e^{-0.2 \cdot 1}(0.2 \cdot 1)^0}{0!}$$

$$= e^{-0.2} = 0.819.$$

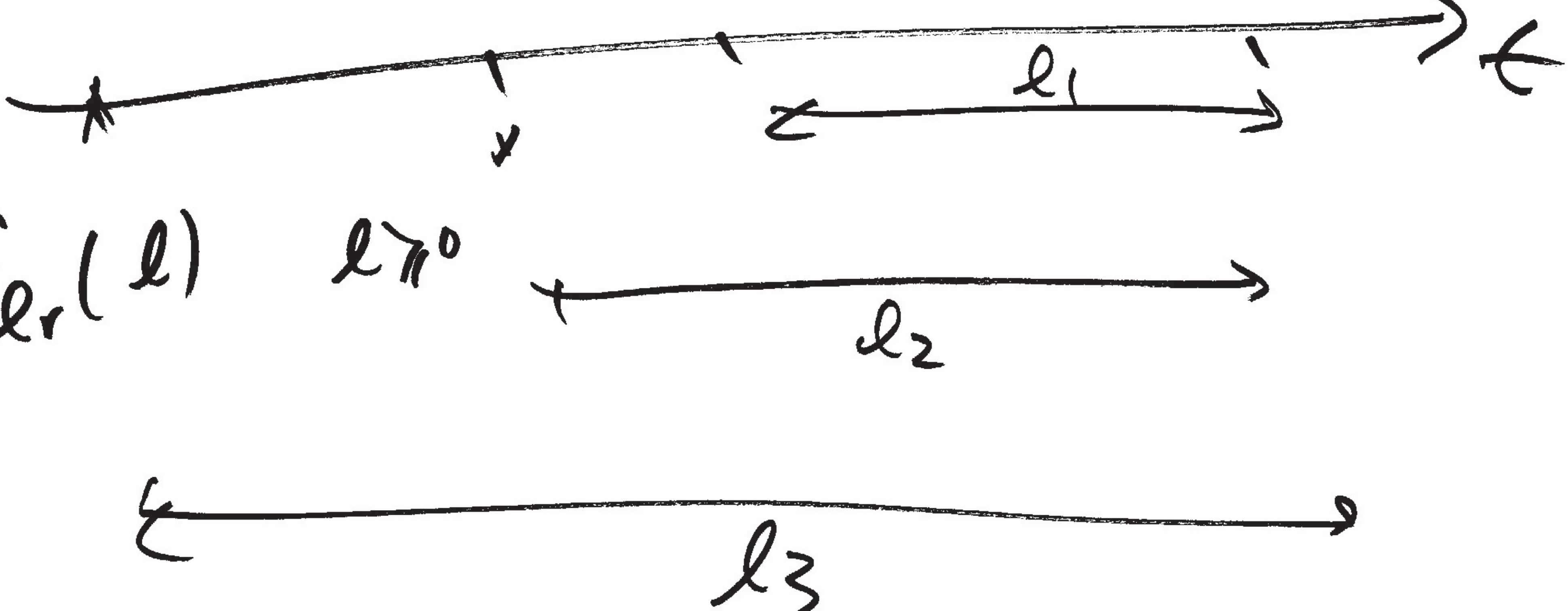
Prob 1 message in 1 hour?

$$P(1,1) = \frac{e^{-0.2 \cdot 1}(0.2 \cdot 1)^1}{1!} = 0.2e^{-0.2}$$

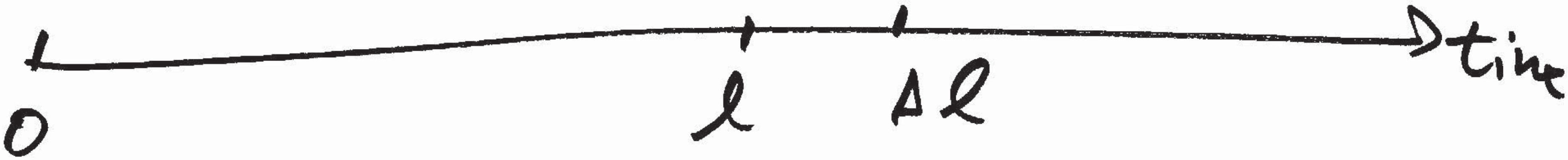
message hour

Interarrival Times for Poisson

l_r = **cont.** R.V. pdr. interval of time between any arrival and rth arrival after it.



$f_{l_r}(l) \quad l \geq 0$



small Δl:

$$\Pr(l \le l_r \le l + \Delta l) = f_{l_r}(l)\,\Delta l \Longrightarrow$$

$$= \underbrace{\text{Prob}(r-1, l)}_{\text{Event A}} \; \underbrace{\lambda\,\Delta l}_{\text{Event B}}$$

A = prob that exactly $r-1$ arrivals in interval of length l.

B = conditional prob rth arrival occurs in the next Δl given exactly $r-1$ was in interval l.

$$f_{\ell_r}(\ell) = \frac{(\lambda\ell)^{r-1} \, e^{-\lambda\ell}}{(r-1)!}$$

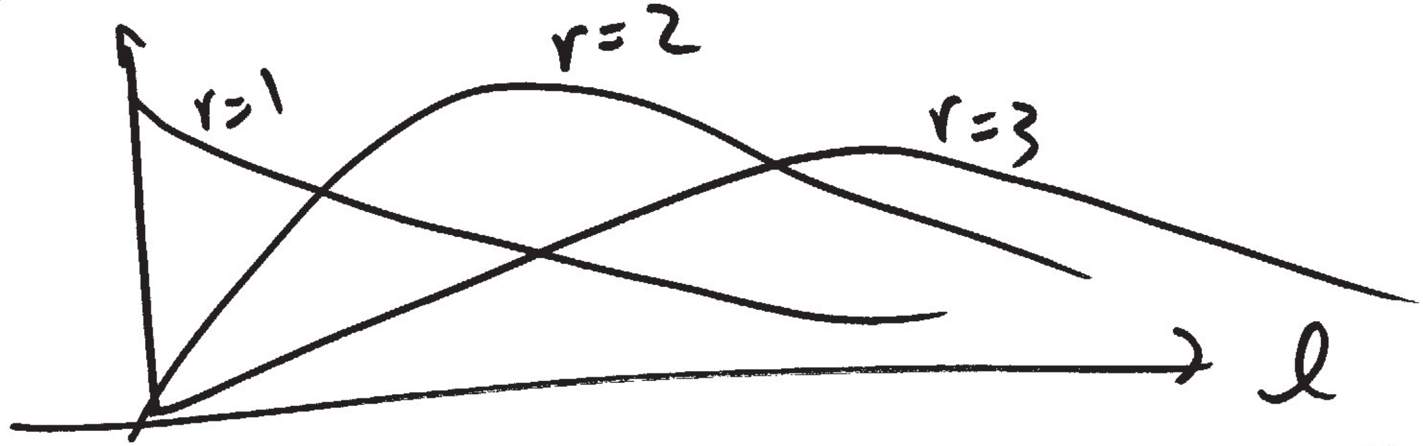
$$f_{\ell_r}(\ell) = \frac{\lambda^{r} \, \ell^{r-1} \, e^{-\lambda\ell}}{(r-1)!} \qquad \ell \geq 0, \quad r = 1, 2, \ldots$$

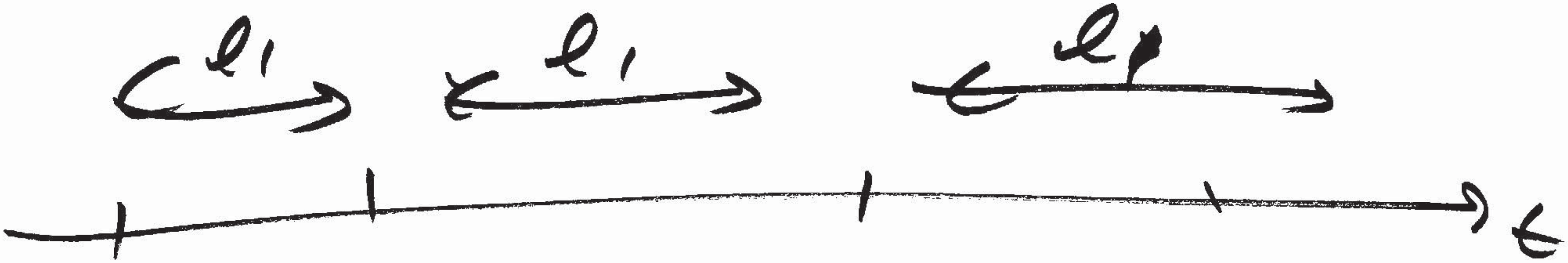
ℓ_r = Erlang R.V. of order r.

$\ell = 1$ $\quad f_{\ell_1}(\ell) = \lambda e^{-\lambda\ell} \qquad \ell \geq 0$

mean $1/\lambda$

Var. $\frac{1}{\lambda^2}$





l_r = sum of r l_1 R.V.

$$M_{l_r}(s) = \left[M_{l_1}(s)\right]^r = \left(\frac{\lambda}{s+\lambda}\right)^r$$

$$E(l_r) = \frac{r}{\lambda} \qquad \sigma^2_{l_r} = \frac{r}{\lambda^2}$$