Renewal Process

Def

1st order interarrival time

Renewal Process: 1st order interarrival times are mutually indep. R.V. described by the same PDF. \( f_x(x_0) \)

Let \( x = \) R.V. Total duration of the present interarrival gap.

Thought exp: Suppose most recent arrival was \( t \) seconds ago. What is the conditional prob of \( x \)?
\[ f(x \mid x > \tau) = \frac{f_x(x_0)}{\int_{\tau}^{\infty} f_x(x_0)dx_0} \]

\[ f(x_0) \]

\[ P_{x \leq \tau} \]

\[ y = \text{remaining time in the present gap till the next arrival.} \]

\[ y = x - \tau \]
The conditional pdf for $y$ given $x$ is:

$$f(y|x) = \frac{\exp(-y) \cdot (y+1)^x}{\Gamma(x+1)}$$

for $y > 0$ and $x > 0$. The marginal pdf is obtained by:

$$\int_0^\infty f(y|x) \, dy = \int_0^\infty \frac{\exp(-y) \cdot y^x}{\Gamma(x+1)} \, dy = 1$$

for $x > 0$.
Ex. born light bulbs one at a time. replace each bulb the instant it fails.

life time of a bulb.

\[
\begin{align*}
 f_x(x) & \begin{cases} 
 1 & 0 < x < 1 \\
 0 & \text{otherwise}
\end{cases} \\
 f_y|_{x \geq 2} & \begin{cases} 
 1 & 1 < x < 2 \\
 0 & \text{otherwise}
\end{cases}
\end{align*}
\]
For Poisson $f_{y | x \geq 2}(y_0 | x \geq 2)$ is indep of $x$ and still an exp.

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**Random Incidence**

Assume renewal process $\rightarrow$ PDF $F_x(t_0)$ 1st order inter arrival time.

$y$ = waiting time until next arrival.

Random Incidence: Pick a random time unit next arrival.
$W =$ total duration of interval gap
into which we enter by random
occurrences.

$p(x)$ (distribution of interval gap)

0 1 2 3 4 5 \ldots 10

second
x and w both represent intercarriol gap.

But for a diff. exp.

\[ x \xrightarrow{\text{exp}} \text{pick a gap.} \]
\[ y \xrightarrow{\text{exp}} \text{pick an ist ance} \]

\[ P_w(w) \]

We are 10 times more likely to fall into a 10 sec. gap than a 1 sec. gap.
$P_{w}(w_0)$ is proportional to $P_x(x_0)$ weighted by $x_0$.

$P_{w}(w_0) \propto w_0 P_x(w_0)$

Proposition:

To normalize to 1, divide by $\sum w_0 P_x(w_0)$

$P_{w}(w_0) = \frac{w_0 P_x(w_0)}{\sum w_0 P_x(w_0)}$
\[ w_0 = 1 \]
\[ P_w(1) = \frac{1 \times \frac{1}{2}}{5.5} = \frac{1}{11} \]
\[ P_w(10) = \frac{10 \times \frac{1}{2}}{5.5} = \frac{10}{11} \]

\[ D_x(w) = 90 \]

\[ E(x) = \frac{90}{10} \cdot \frac{1}{10} = 1.9 \]

\[ P_w(w_0) = \begin{cases} 
1 \times 0.9 & = 0.9 \\
\frac{10 \times 0.1}{1.9} & = \frac{1}{1.9} \\
\frac{10 \times 0.1}{1.9} & = \frac{1}{1.9} \\
\end{cases} \]

\[ w_0 = 1 \]
\[ w_0 = 10 \]
End goal: $f_{y\mid w}(y_0/w_0) = \begin{cases} \frac{1}{w_0} & 0 \leq y_0 \leq w_0 \\ 0 & \text{otherwise} \end{cases}$

Goal: $f_y(y_0)$.

$f_{w\mid (w_0)} = \frac{w_0 f_x(w_0)}{E(x)}$
\[ f_{w,y}(w_0, y_0) = f_w(w_0) \cdot f_{y|w}(y_0|w_0) \]

\[ f_{w,y}(w_0, y_0) = \frac{w_0 f_x(w_0)}{E(X)} \cdot \frac{1}{w_0} \]

\[ 0 \leq y_0 \leq w_0 \leq \infty \]

\[ f_{y|w}(y_0) = \int f_{w,y}(w_0, y_0) \, dw_0 \]

\[ = \int_{w_0=y_0}^{w_0} \frac{w_0 f_x(w_0)}{E(X)} \, dw_0 = \]

\[ = \int_{w_0=y_0}^{w_0} \]
\[ f_{y_i}(y_0) = \frac{1}{E(x)} \left( 1 - P_{x \leq (y_0)} \right) \]
Q: \( f_w(w) \)? for the bulb problem?

\[
f_w(w) = \frac{w_0 f_x(w)}{E(x)}
\]

\[
f_w(w) = 2w_0 f_x(w)
\]

Q: What is \( f_y(y_0) \)?
\[ f(y_0) = \frac{1}{E(x)} \left( 1 - P_{x \leq y_0} \right) \]
\[
1 - P_x \leq \frac{\mathbb{E}(y)}{\mathbb{E}(x)} = f_y(y_0)
\]

- Picking a y.
- Exp. Random Incident pick an instance.