Renewal Process

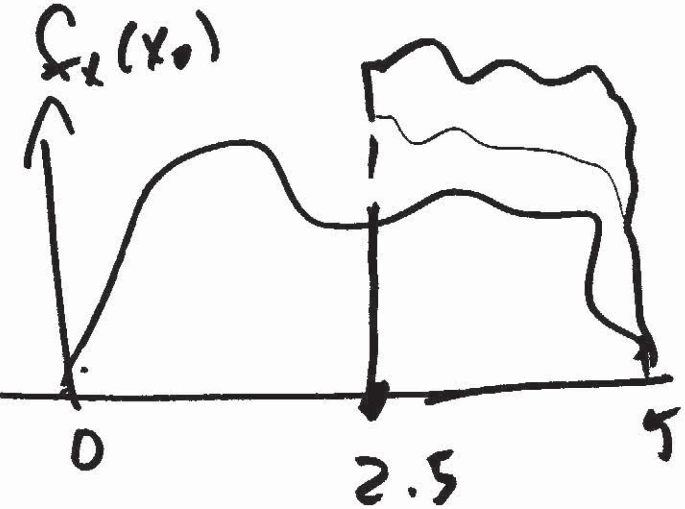
Def

Renewal Process : 1st order interarrival Times are mutually indep. R.V. described by the same pdf. — $f_x(x_0)$

Let x = R.V. Total duration of the present interarrival gap.

Thought exp : Suppose most recent arrival was τ seconds ago. What is the conditional prob of x?

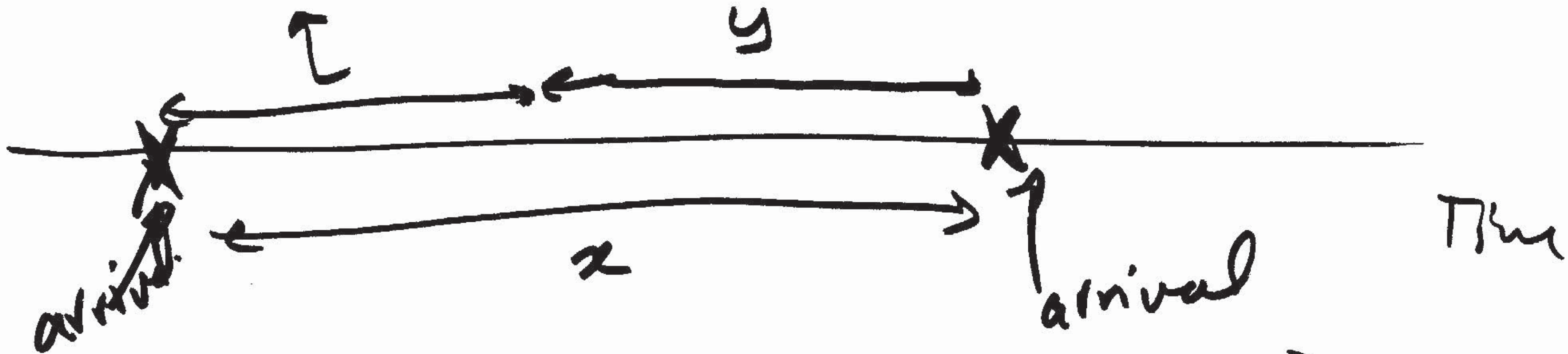
$$f_{X|X>\tau}(x_0 \mid X > \tau) = \frac{f_X(x_0)}{\int_{\tau}^{\infty} f_X(x_0)\,dx_0}$$



$$= \frac{f_X(x_0)}{1 - P_{X\le}(\tau)}$$

y = remaining time in the present gap till the next arrival.

$y = x - \tau$

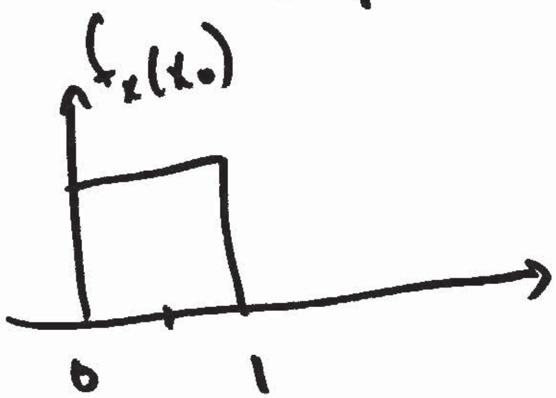


what is the conditional pdf for y?

$$f_{y|x>\tau}(y_0 \mid x>\tau) = \frac{f_y(y_0)}{\int_{\tau}^{\infty} f_x(x_0)\,dx_0} = \frac{f_x(y_0+\tau)}{\int_{\tau}^{\infty} f_x(x_0)\,dx_0}$$

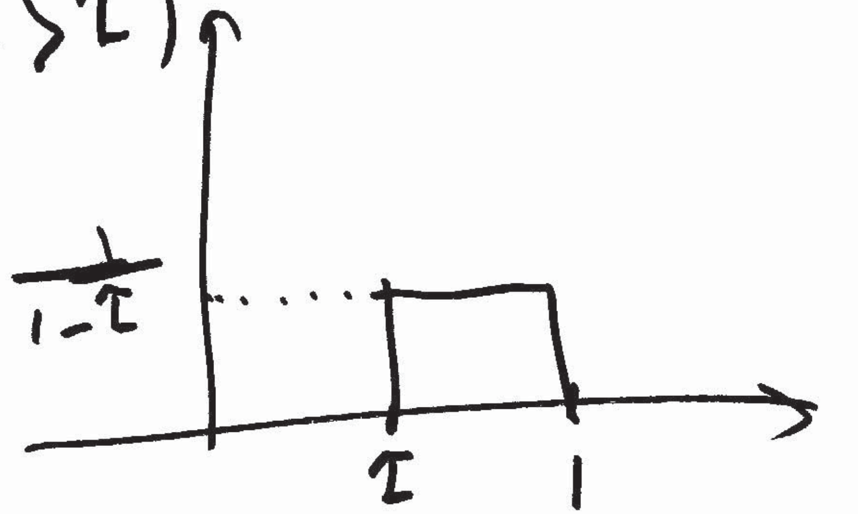
$$= \frac{f_x(y_0+\tau)}{1 - P_{x\leq}(\tau)} \qquad y_0 > 0$$

Ex burn light bulbs one at a Time. replace each bulb The instant it fails.

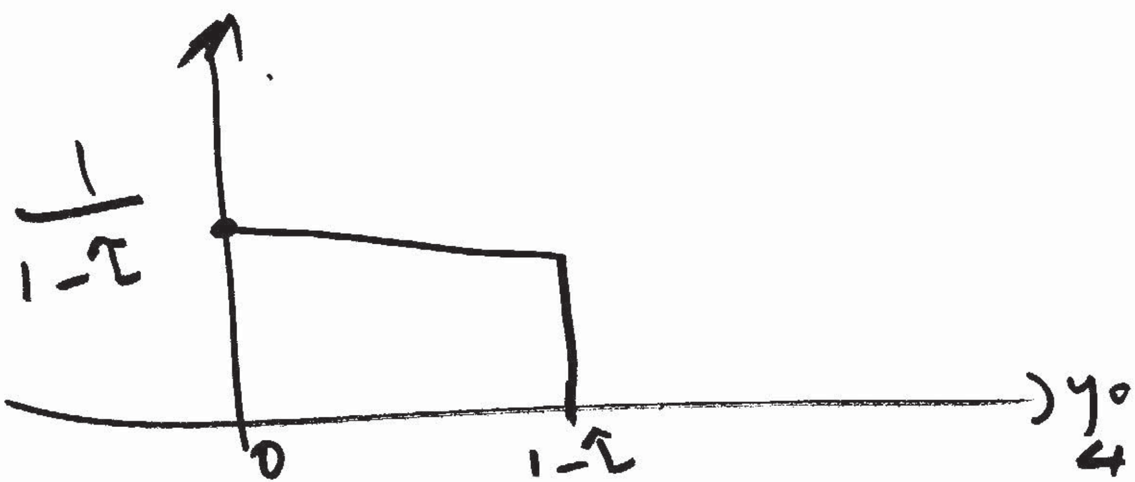


life Time of a bulb.

~~[illegible]~~ $f_{X|X>\tau}(x_0 \mid X>\tau)$



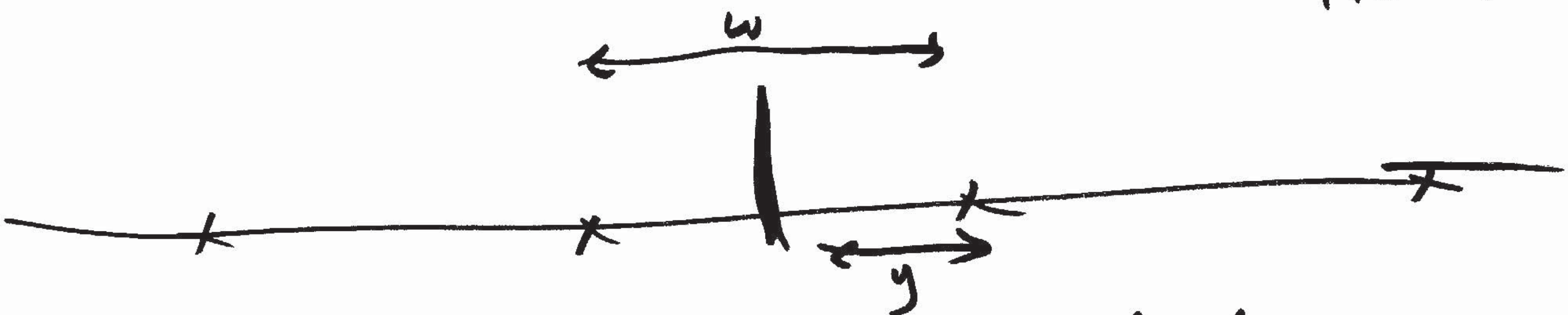
$f_{Y|X>\tau}(y_0 \mid X>\tau)$



For Poisson $f_{Y|X>\tau}(y_0 / x > \tau)$ is indp of τ and still an exp.

Random Incidence

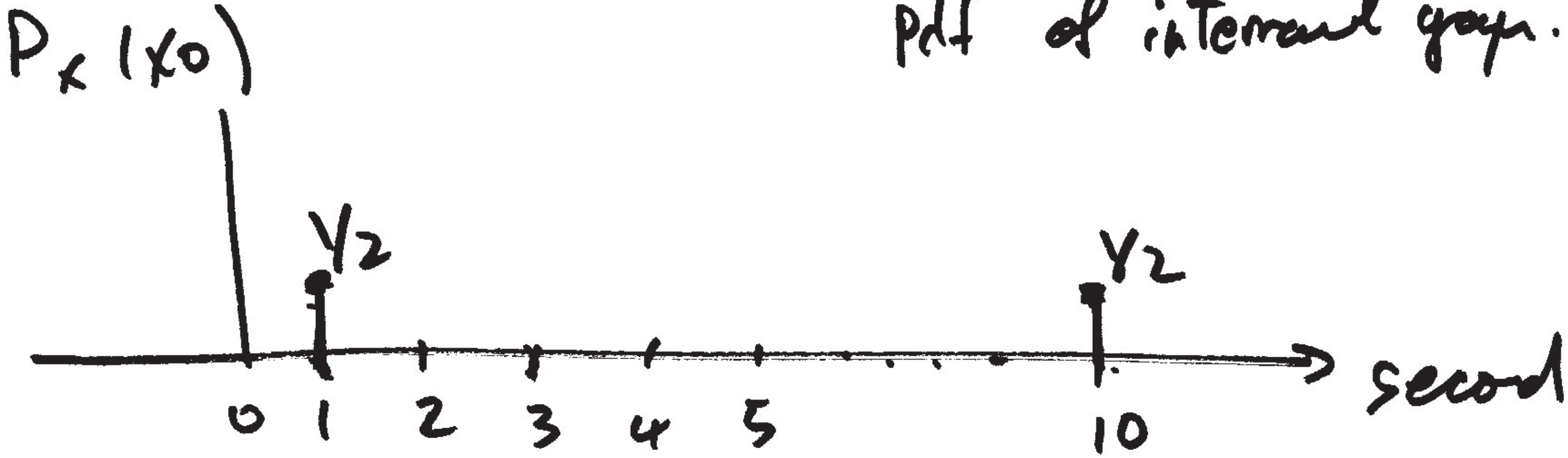
Assume Renewal process → Pdf. $f_x(x_0)$ 1st order inter arrive Time.

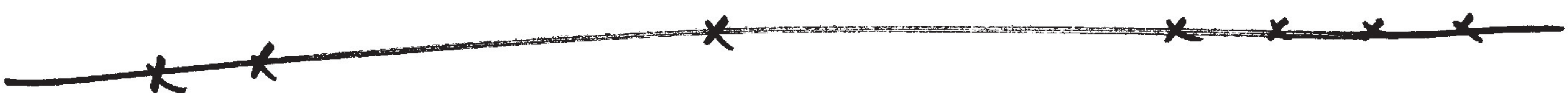


y = waiting time unti next arrival.

Random Incidence: Pick a random time unit next arrival.

W = Total duration of interarrival gap into which we enter by random Incidence.



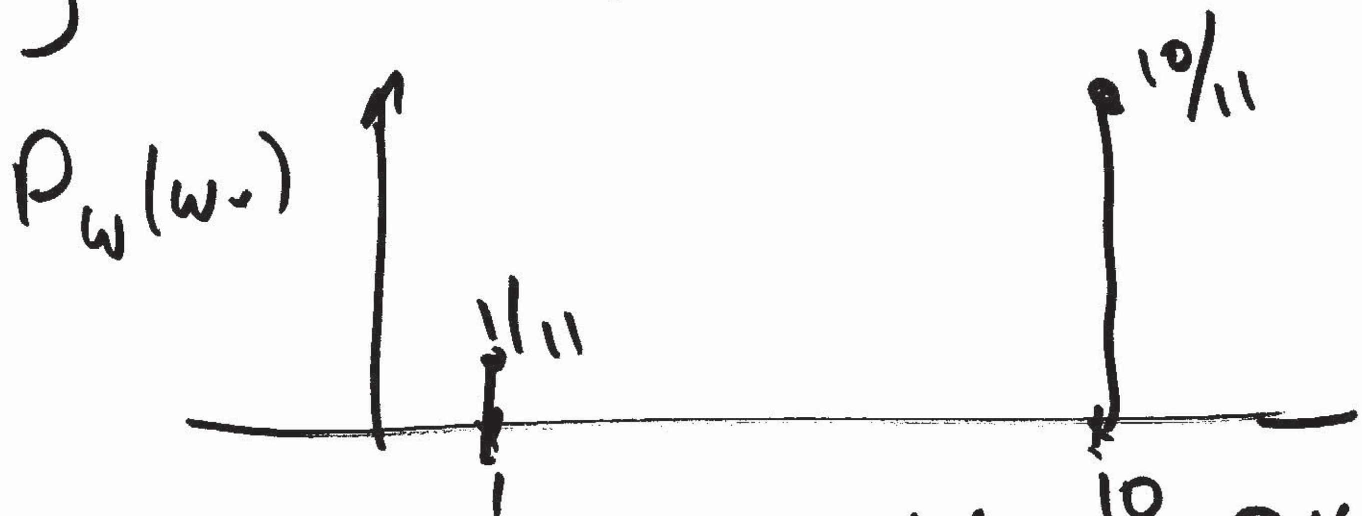


x and w both represent interarrival gap.

But for 2 diff. exp.

$x \xrightarrow{exp} $ pick a gap.

$y \xrightarrow{exp} $ pick an instance



We are 10 times more likely to fall into a 10 sec. gap than a 1 sec. gap.

$P_W(w_0)$ is proportional to $P_x(x_0)$ weighted by x_0

$$P_W(w_0) \propto w_0 P_x(w_0)$$

↑ proportion

to normalize to 1 divide by $\int w_0 P_x(w_0)$

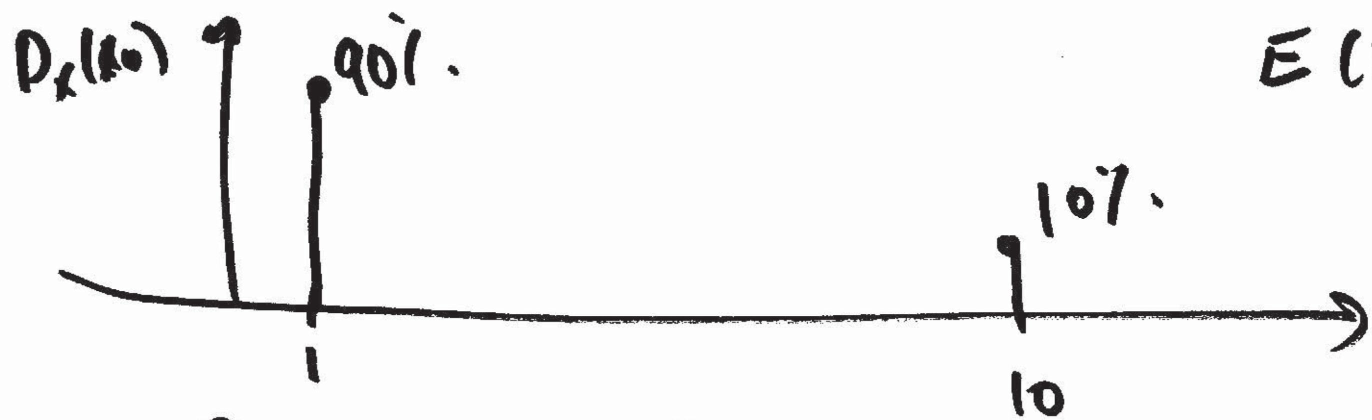
pdf → $$P_W(w_0) = \frac{w_0 P_x(w_0)}{\int w_0 P_x(w_0)\, dw_0} = \frac{w_0 P_x(w_0)}{E(x)}$$

discrete case

pmf → $$P_W(w_0) = \frac{w_0 P_x(w_0)}{\sum w_0 P_x(w_0)}$$

$w_0 = 1$ $\quad P_w(1) = \dfrac{1 \times \frac{1}{2}}{5.5} = 1/11$

$$P_w(10) = \frac{10 \times \frac{1}{2}}{5.5} = \frac{10}{11}$$



$E(x) = \frac{9}{10}\cdot 1 + 10 \times \frac{1}{10} = 1.9$

$$P_w(w_0) = \begin{cases} \dfrac{1 \times 0.9}{1.9} = \dfrac{0.9}{1.9} & w_0 = 1 \\[2ex] \dfrac{10 \times 0.1}{1.9} = \dfrac{1}{1.9} & w_0 = 10 \end{cases}$$

$$f_W(w_0) = \frac{w_0 P_X(w_0)}{E(X)}$$

End goal: $f_Y(y_0)$:

Cont.

$$f_W(w_0) = \frac{w_0 f_X(w_0)}{E(X)}$$

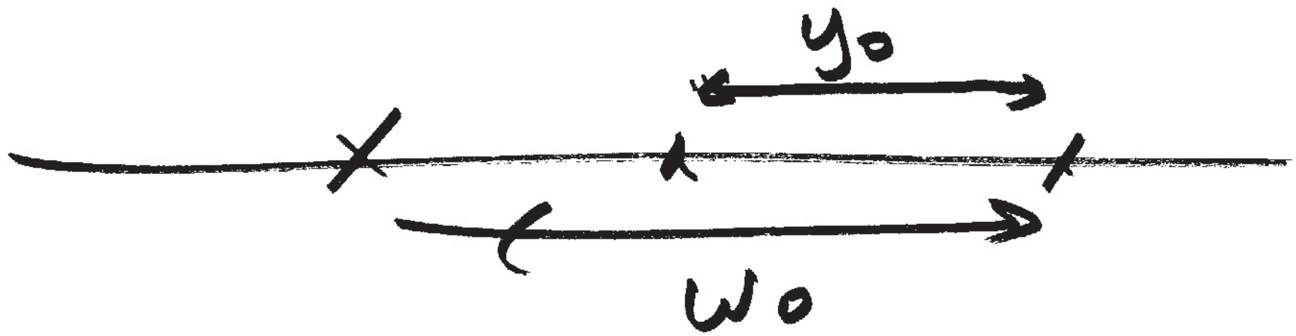
goal: $f_Y(y_0)$.

$$f_{Y|W}(y_0 | w_0) = \begin{cases} \frac{1}{w_0} & 0 \le y_0 < w_0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{W,Y}(w_0,y_0) = f_W(w_0)\cdot f_{Y|W}(y_0|w_0)$$

$$f_{W,Y}(w_0,y_0) = \frac{w_0\, f_X(w_0)}{E(X)} \cdot \frac{1}{w_0}$$

$$0 \le y_0 \le w_0 < \infty$$



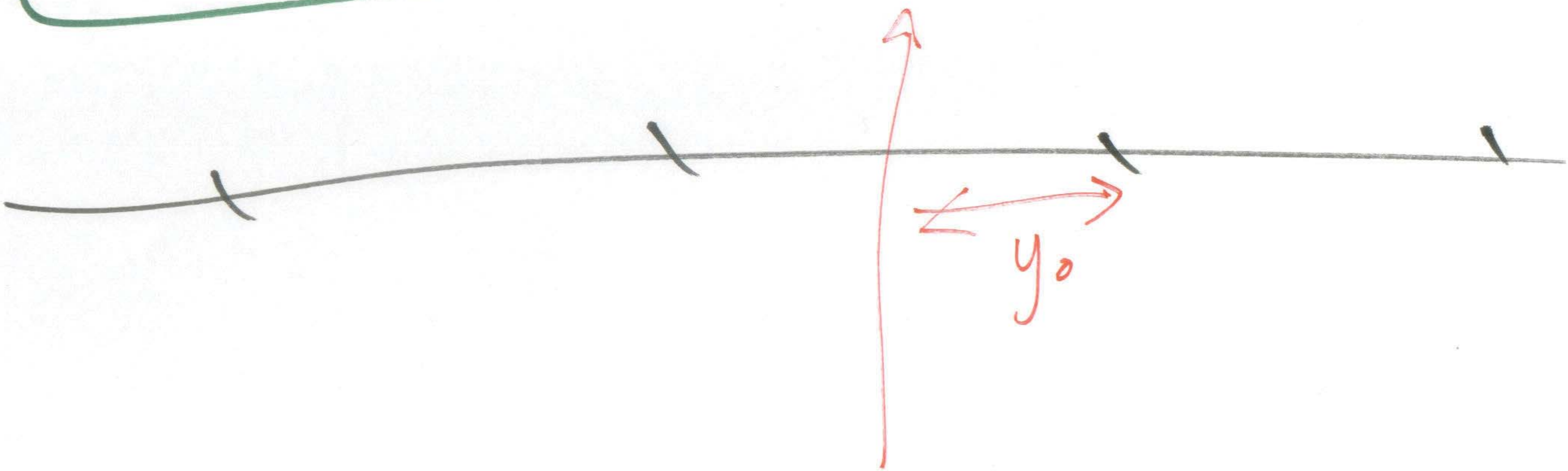
$$f_Y(y_0) = \int_{w_0} f_{W,Y}(w_0,y_0)\, dw_0$$

$$= \int_{w_0=y_0}^{\infty} \frac{f_X(w_0)}{E(X)}\, dw_0 =$$

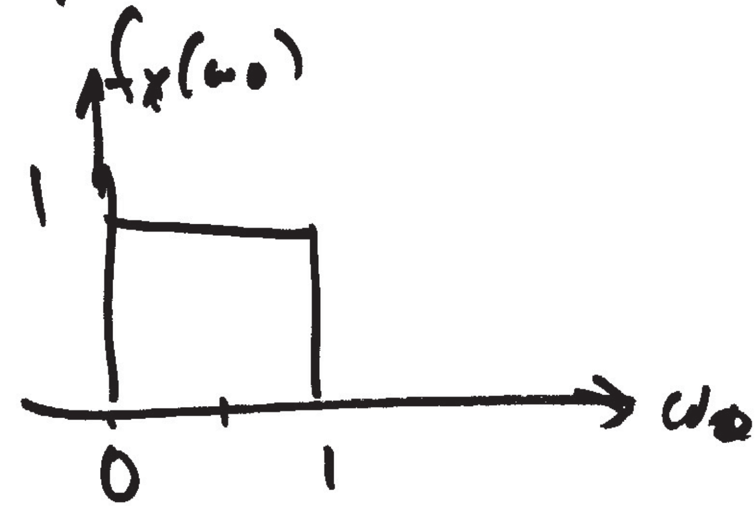
//

$$= \frac{1}{E(x)} \int_{w_0=y_0}^{\infty} f_x(w_0)\, dw_0$$

$$f_y(y_0) = \frac{1}{E(x)} \left(1 - P_{x\leq}(y_0)\right)$$



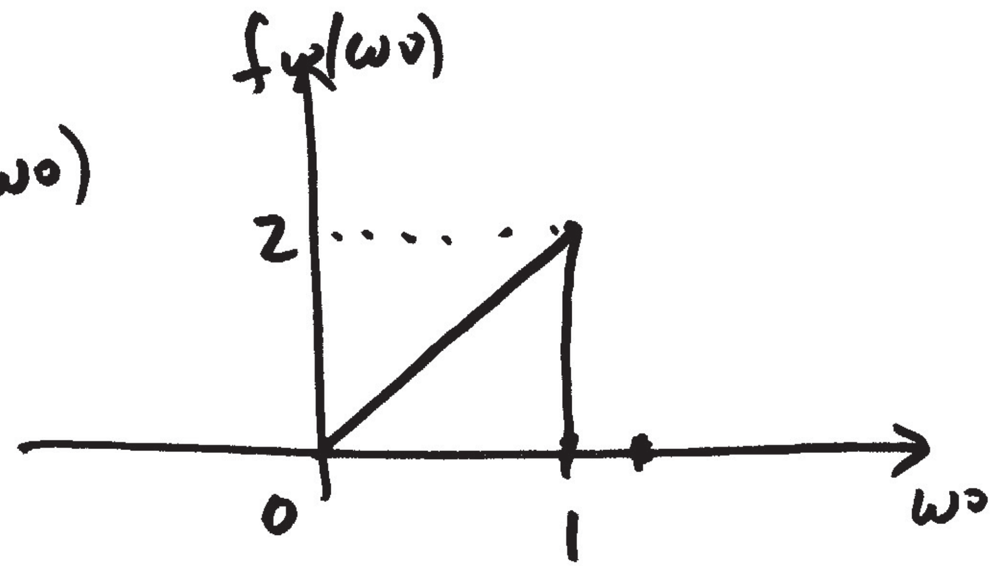
Q $f_W(w_0)$ for the bulb problem?



$$f_W(w_0) = \frac{w_0 f_X(w_0)}{E(X)}$$

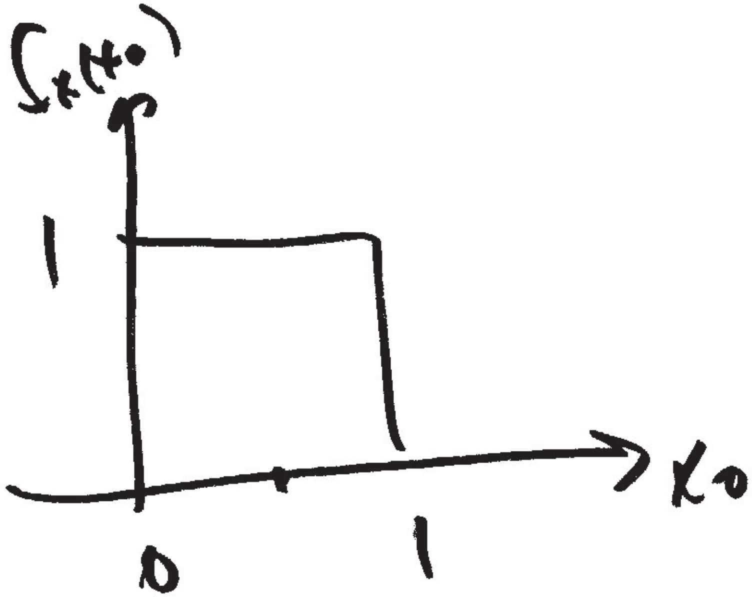
$$= \frac{w_0 f_X(w_0)}{1/2}$$

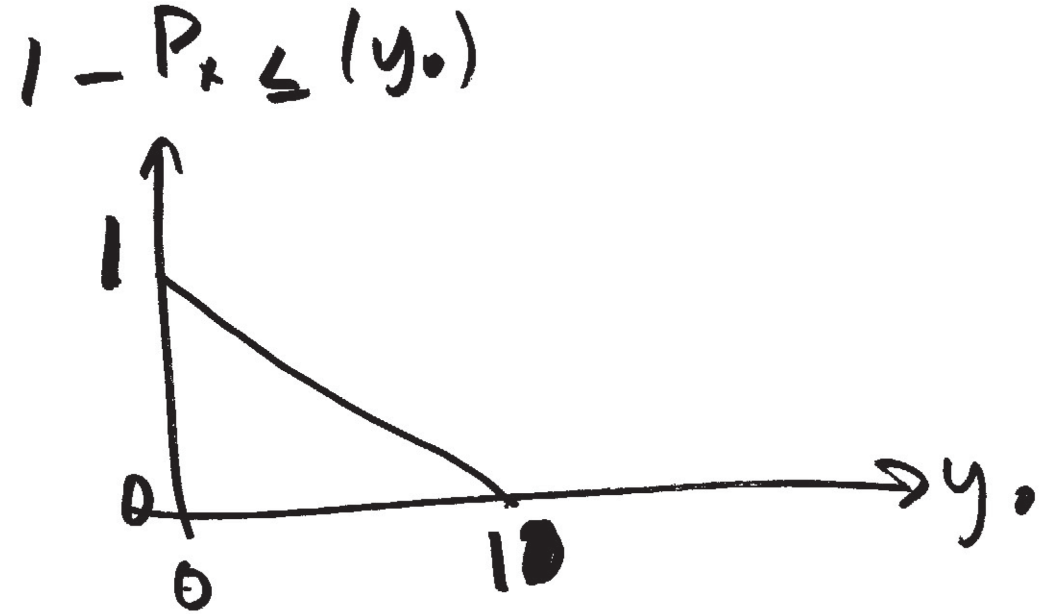
$$f_W(w_0) = 2 w_0 f_X(w_0)$$



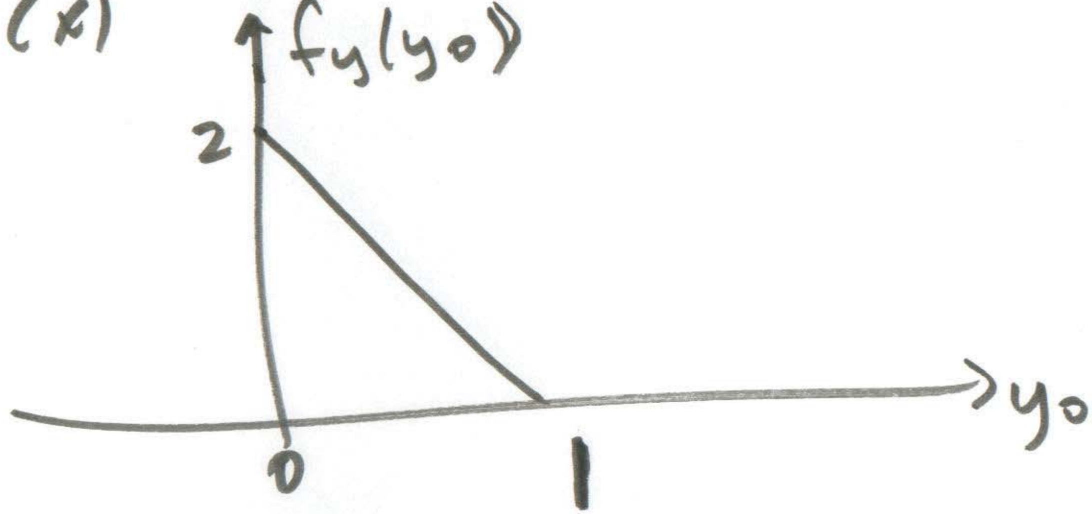
Q What is $f_Y(y_0)$?

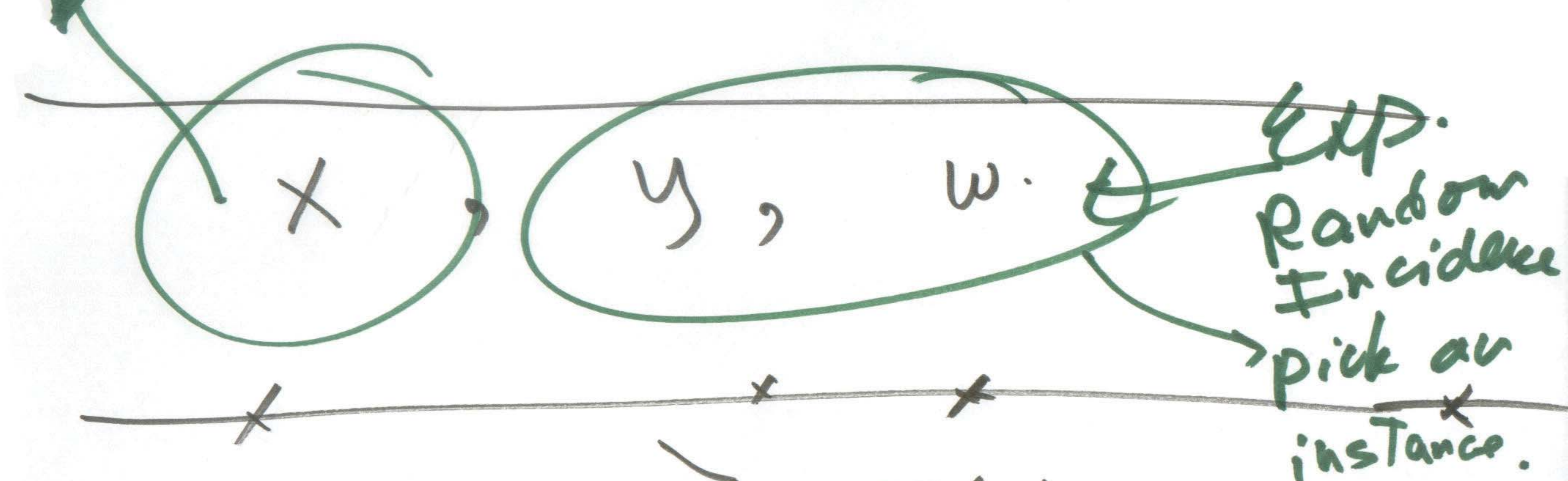
$$f_Y(y_0) = \frac{1}{E(X)} \left(1 - P_{X\leq}(y_0)\right)$$





$$\frac{1 - P_x \leq (y_0)}{E(x)} = f_y(y_0)$$





$$E(y) \gtreqless E(x)$$