Fundamental limit theorem

Markov Inequality:
Suppose R.V. $X$ taken on a non-negative value.
then: $P(X \geq a) \leq \frac{E(X)}{a}$

Proof:
Define $Y_a = \begin{cases} a & \text{if } x \leq a \\ 0 & \text{if } x > a \end{cases}$
\[ Y_a \leq X \implies E[Y_a] \leq E(X) \]

Compute \( E(Y_a) = a \cdot P[Y_a = a] + 0 \cdot P[Y_a = 0] \)

\[ = a \cdot P[Y_a = a] = a \cdot P(X \geq a) \]

\[ \implies P(X \geq a) \leq \frac{E(X)}{a} \]

Ex: R.V. X. uniform \([0, 4]\) \& \( E(X) = 2 \)

Markov Inequality: \( a = 2 \)

\[ P(X \geq \sqrt{2}) \leq \frac{2}{2} = 1 \]

\( \iff \) useless
\[
\begin{align*}
\alpha = 3 \Rightarrow \quad P(X \geq 3) & \leq \frac{2}{3} \\
a = 4 \Rightarrow \quad P(X \geq 4) & \leq \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
P(X > 2) & = \frac{1}{2} \\
P(X > 3) & = \frac{1}{4} \\
P(X > 4) & = 0
\end{align*}
\]

Note: Chebychev Inequality
$X$ is R.V. mean $\mu$, variance $\sigma^2$

$$P\left( |X-\mu| > C \right) \leq \frac{\sigma^2}{C^2} \quad \forall C > 0$$

Proof: Consider $(X-\mu)^2$ as a positive R.V.

Apply Markov inequality with $\alpha = C^2$

$$P\left( (X-\mu)^2 > C^2 \right) \leq \frac{E[(X-\mu)^2]}{C^2}$$

Note: event $(X-\mu)^2 > C^2$ is identical to $|X-\mu| > C$
$\Pr\left( |X-\mu| > c \right) \leq \frac{\sigma^2}{c^2}$ \quad \forall c > 0$

**Chebyshev Inequality**

let \( c = k \sigma \) \( k > 0 \), positive

$\Pr\left( |X-\mu| > k \sigma \right) \leq \frac{\sigma^2}{k^2 \sigma^2} = \frac{1}{k^2}$

**Words**

prob that a R.V. $X$ is more than \( k \) standard deviation away from mean is at most $\frac{1}{k^2}$
Ex R.V. uniform \([0, 4]\) \(E(x) = 2\), \(\sigma^2 = \frac{4}{3}\)

\[P(|x - 2| > 1) \leq \frac{4}{3} \rightarrow \text{useless.}\]

Ex \(X\) is exponential \(\lambda \neq 1\)

\(E(X) = \text{Var}(X) = 1\).

\[P(X > c) = P(X - 1 > c - 1) \leq \frac{6^2}{(c-1)^2(c+1)^2} \]

\[P(X > c) \leq \frac{1}{(c-1)^2} \leftarrow \text{Chebyshev.}\]

\[\text{loose bound}\]

\(\text{using exponential}\)

\[P(X > c) = e^{-c}\]
Weak Law of Large #5

Consider \( X_1, X_2, \ldots \) i.i.d. RV. mean \( \mu \), var \( \sigma^2 \)

Define \( \bar{M}_n = \frac{X_1 + X_2 + \ldots + X_n}{n} \)

\( \text{"Sample mean"} \)

\[ E[\bar{M}_n] = \frac{E[X_1] + E[X_2] + \ldots + E[X_n]}{n} = \frac{n\mu}{n} = \mu \]

\[ \text{Var}[\bar{M}_n] = \frac{n\sigma^2}{n} = \frac{\sigma^2}{n} \]

Apply Chebyshev's To \( \bar{M}_n \).
\[ P\left( \left| \overline{X}_n - \mu \right| \geq \varepsilon \right) \leq \frac{ \sigma^2 }{ n \varepsilon^2 } \quad \forall \varepsilon > 0 \]

\( \lim_{n \to \infty} \) of both sides.

\[ P\left( \left| \overline{X}_n - \mu \right| \geq \varepsilon \right) \to 0 \quad \text{as} \quad n \to \infty \]

**Weak Law of Large Numbers.**

**EX** Event A defined in terms of probabilistic exp.

\[ P(A) = P \]

\( n \) independent repetition of the experiment.

\( X_i \) \( i^{th} \) repetition of the experiment event A occurs

\[ X_i = \begin{cases} 1 & \text{event A occurs} \\ 0 & \text{otherwise} \end{cases} \]
\[ M_n = \frac{X_1 + X_2 + \ldots + X_n}{n} \] measured quantity

\[ E[X_i] = 1 \cdot P(A) + 0 \cdot P(\bar{A}) = P(A) = p. \]

Apply Weak Law of Large Numbers:

\[ P\left(\left|\frac{M_n - p}{\sqrt{n}}\right| \geq \epsilon\right) \to 0 \text{ as } n \to \infty \]

Empirical frequency is a good way to estimate \( p \).
Ex polling: $p = \text{probability of supporting Kerry.}$

Interview $n$ randomly selected person

Record $M_n$.

Reply of the $n$ persons is an independent Bernoulli R.V. $X_i$ with success prob $p$.

Var $6^2 = p(1-p)$

Mean $= p$.

Apply Chebyshev:

$P \left( |M_n - p| \geq \varepsilon \right) \leq \frac{p(1-p)}{n\varepsilon^2}$

$
\begin{cases}
\frac{1}{4} 
\end{cases}
$

Note: $p(1-p) \leq \frac{1}{4}$
\[ P \left( \left| M_n - p \right| \geq \varepsilon \right) \leq \frac{1}{4n\varepsilon^2} \]

\[ \text{Ex} \quad \varepsilon = 0.1 \quad n = 100 \]

\[ P \left( \left| M_{100} - p \right| \geq 0.1 \right) \leq \frac{1}{4} \]

Chances that our \( M_{100} \) estimate of \( p \) is off by more than 10\% is less than \( \frac{1}{4} \)
Our estimate needs to be within 1% of $p$ with prob of at least 95%.

$$P\left( |M_n - p| > 0.01 \right) \leq 5\%$$

$$P\left( |M_n - p| > 0.01 \right) \leq \frac{1}{4n(0.01)^2} \leq 5\%$$

$\Rightarrow n \geq 50,000$