Fundamental limit theorem

Markov Inequality:

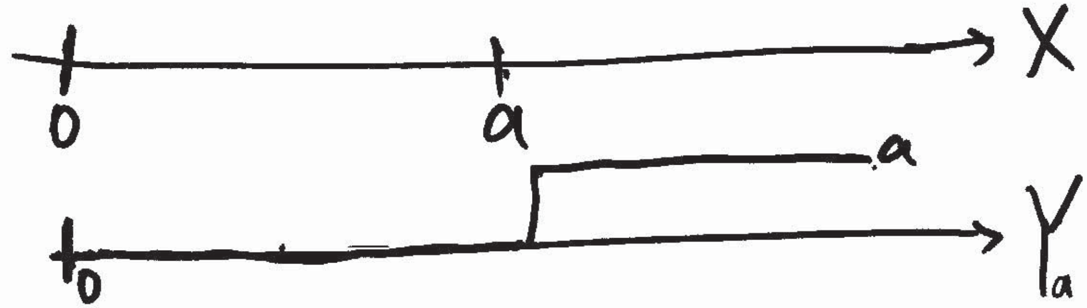
Suppose R.V. X takes on a ~~nonnegative~~ positive value. $\forall a > 0$

Then: $P(X \geq a) \leq \dfrac{E(X)}{a}$

Markov Inequality.

Proof: Define.

fix a. $\longrightarrow$ $Y_a = \begin{cases} 0 & X < a \\ a & X \geq a \end{cases}$



$$Y_a \le X \implies E[Y_a] \le E(X)$$

Compute $E(Y_a) = a\, P[Y_a = a] + 0 \cdot P[Y_a = 0]$

$$= a\, P[Y_a = a] = a\, P(X \geq a)$$

$$\implies a\, P(X \geq a) \le E(X)$$

$$\implies \boxed{P(X \geq a) \le \frac{E(X)}{a}}$$

Markov

Ex R.V. X. uniform $[0, 4]$; $E(X) = 2$

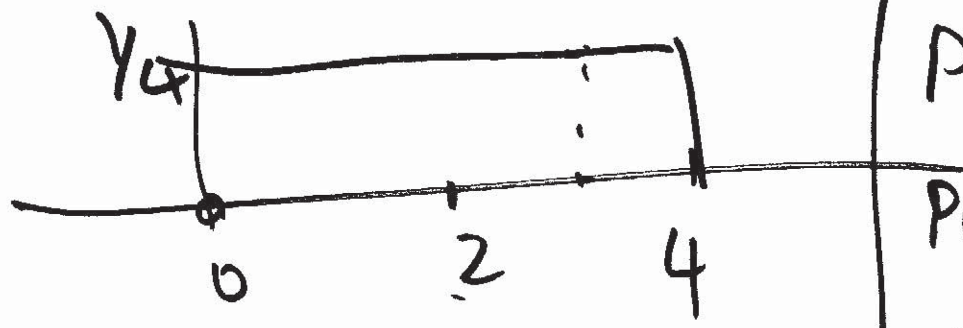
Markov Inequality

$a = 2$

$$P(X \geq 2) \le \frac{2}{2} = 1 \implies \text{useless}$$

$a = 3 \Rightarrow \quad P(X \geq 3) \leq \frac{2}{3}$

$a = 4 \Rightarrow \quad P(X \geq 4) \leq \frac{1}{2}$



$P(X > 2) = \frac{1}{2}$

$P(X > 3) = \frac{1}{4}$

$P(X > 4) = 0$

Precise.

Chebychev Inequality

X is R.V. mean $\underline{\underline{\mu}}$, variance $\underline{\underline{\sigma^2}}$

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \qquad \forall c > 0$$

<u>Proof</u>: Consider $(X-\mu)^2$ as a positive R.V.

Apply Markov inequality with $a = c^2$

$$P((X-\mu)^2 \geq c^2) \leq \frac{E[(X-\mu)^2]}{c^2} = \frac{\sigma^2}{c^2}$$

<u>Note</u>: event $(X-\mu)^2 \geq c^2$ is identical to $|X-\mu| \geq c$

$$\boxed{P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \quad \forall c > 0}$$

Chebychev Inequality.

let $c = k\sigma$ k positive

$$\boxed{P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}}$$

Words: prob That a R.V. ~~Takes on~~ is more than k standard deviation away from mean is at most $\frac{1}{k^2}$

Ex R.V. uniform $[0,4]$ $E(X)=2$, $\sigma^2 = 4/3$

$$P(|X-2| \geq 1) \leq \frac{4}{3} \longrightarrow \text{useless.}$$

Ex X is exponential R.V. $\lambda=1$

$E(X) = Var(X) = 1.$ $c \geq 1$

$$P(X > c) = P(X - 1 > c - 1) \leq \frac{\sigma^2}{(c-1)^2} = \frac{1}{(c-1)^2}$$

$$P(X > c) \leq \frac{1}{(c-1)^2} \leftarrow \text{chebyshev.}$$

$\longrightarrow$ loose bound

$\longrightarrow$ using exponent $P(X \geq c) = e^{-c}$

Weak Law of Large #s

Consider $X_1, X_2 \ldots$ I.I.D. R.V. mean μ var σ^2

Define $M_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

"sample mean"

$$E[M_n] = \frac{E[X_1] + E[X_2] + \cdots E[X_n]}{n} = \frac{n\mu}{n} = \mu$$

$$\mathrm{Var}[M_n] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Apply Chebychev To M_n:

$$P(|M_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \qquad \forall \epsilon > 0$$

lim as $n \to \infty$ of both sides.

$$\boxed{P(|M_n - \mu| \geq \epsilon) \to 0 \quad \text{as } n \to \infty}$$

weak law of large #.

<u>Ex</u> Event A defined in terms of probabilistic exp.

$P(A) = p$

n independent repetition of the experiment.

ith repetition X_i.

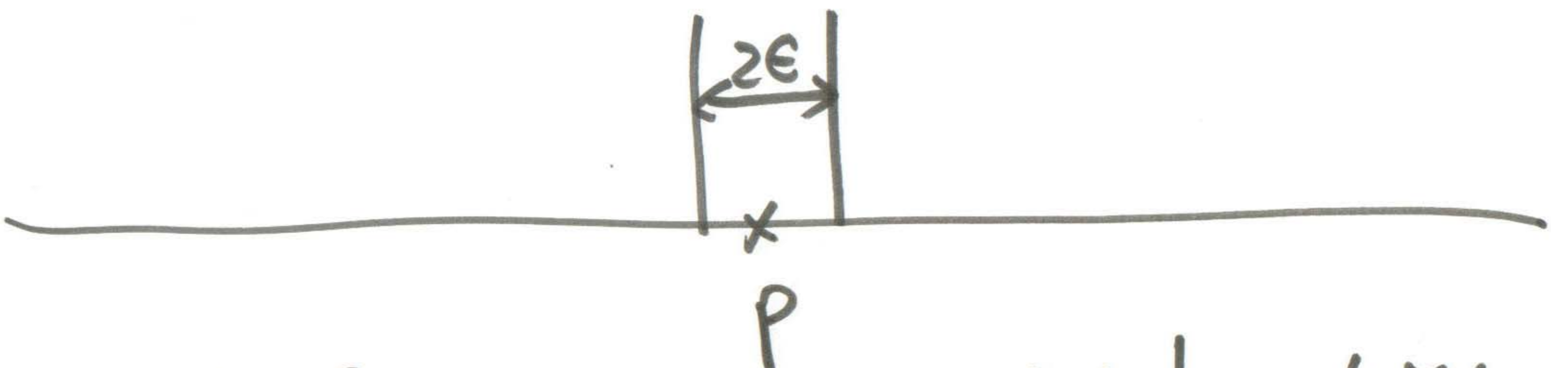
$$X_i = \begin{cases} 1 & \text{event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$ ← measured quantity

$$E[X_i] = 1 \cdot P(A) + 0 \cdot P(\bar{A}) = P(A) = p.$$

Apply Weak law of large #s:

$$P(|M_n - p| \geq \epsilon) \to 0 \quad \text{as } n \to \infty \quad \forall\, \epsilon > 0$$



Empirical frequency is a <u>good</u> way to estimate p.

Ex polling: p = probability of supporting Kerry.

interview n randomly selected person

Record M_n.

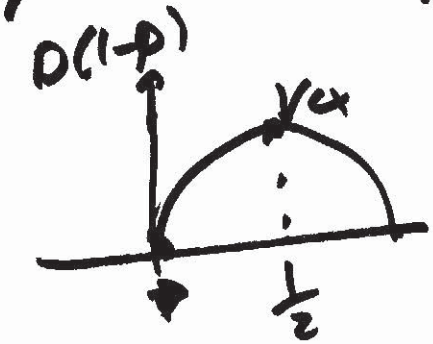
Reply of the n person. is an independent Bernoulli R.V. X_i with success prob of $\underline{\underline{p}}$???

variance $\sigma^2 = p(1-p)$
mean $= p$.

Apply Chebychev:

$$P(|M_n - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2} \Longrightarrow$$

note: $\boxed{p(1-p) \leq \frac{1}{4}}$



$$P(|M_n - p| \geq \epsilon) \leq \frac{1}{4n\epsilon^2}$$

Ex $\epsilon = 0.1$ $n = 100$

$$P(|M_{100} - p| \geq 0.1) \leq \frac{1}{4}$$

chance that our M_{100} estimate of p is off by more than 10% is less than $\frac{1}{4}$.

Ex our estime needs to be within 1%
of p with prob of at least 95%

$$P(|M_n - p| > 0.01) \leq 5\%$$

$$P(|M_n - p| > 0.01) \leq \frac{1}{4n(0.01)^2} \leq 5\%$$

chebyshev

$$\Longrightarrow n \geq 50{,}000$$