Ex of CLT → Binomial distributi..

Ex 16 Bernoulli Trials $P = 0.5$.

Compute prob(# of successes) is 6, 7 or 8.

Exact: $P(6 \leq K \leq 8) = \sum_{k_0=6}^{8} \binom{16}{k_0} P^{k_0} (1-P)^{16-k_0}$

$= 0.49.$

Method 1: (Bad CLT Approx).

$Pr(6 \leq K \leq 8) \simeq \Phi\left(\frac{8-8}{2}\right) - \Phi\left(\frac{6-8}{2}\right)$

$\simeq 0.34$

Method 2 (good CLT approx) $= \Phi\left(\frac{8\frac{1}{2}-8}{2}\right) - \Phi\left(\frac{6-\frac{1}{2}-8}{2}\right)$

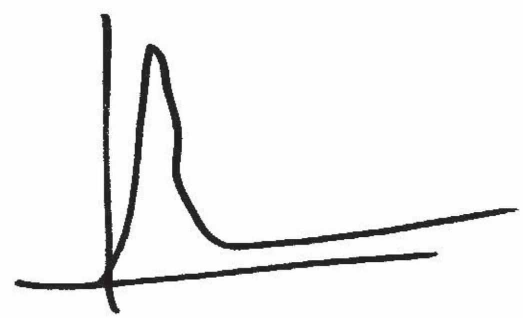
$= 0.49306$

much closer to exact.

When does Gaussian Approx to Binomial fail

If (p) is too close to 0 or to 1

PMF Binomial → too asymmetric



⇒ Much larger n for approx to hold.

Poisson Approx To Binomial

- Recall. Demoivre-Laplace approx no good
  $P \to 0$ $P \to 1$

- what approx does work?

$$P_K(K_0) = \binom{n}{K_0} p^{K_0} (1-P)^{n-K_0}$$

$n \to \infty$ $P \to 0$ $nP = \mu$

$$P_K(K_0) = \frac{n(n-1)\cdots(n-K_0+1)}{K_0!} \left(\frac{\mu}{n}\right)^{K_0} \left(1-\frac{\mu}{n}\right)^{n-K_0}$$

$$= \frac{n(n-1)\cdots(n-K_0+1)}{n^{K_0}} \frac{\mu^{K_0}}{K_0!} \left(1-\frac{\mu}{n}\right)^{n-K_0}$$

take limit $n \to \infty$

$$\lim_{n\to\infty} P_K(k_0) = \frac{\mu^{k_0}}{k_0!} e^{-\mu} \quad \leftarrow \text{poisson.}$$

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^{a}. \quad \leftarrow \text{Recall.}$$

$P \to 0 \quad nP = \mu$

<u>Words</u>: as $n \to \infty$

Binomial $\longrightarrow$ poisson.

$$\lim_{n\to\infty} P_K(k_0) = \frac{(nP)^{k_0} e^{-(nP)}}{k_0!}$$

Ex $n = 100$ $p = 0.01$

	$k_0 = 0$	$k_0 = 1$	$k_0 = 3$	$k_0 = 10$
Exact value $P_k(k_0)$	0.3660	0.3697	.0610	7×10^{-8}
Poisson approx	.3679	0.3679	.0613	10×10^{-8}
DeMoivre-Laplace approx.	0.2420	0.3850	.004	$< 2 \times 10^{-8}$

Proof for CLT

① $X_1, X_2, \ldots, X_n$ iid.
mean $E(X)$ σ_X^2 = finite.

Define $r = X_1 + X_2 + \cdots + X_n$

→ show CDF $P_{r<}(r_0)$ approaches CDF of a Gaussian R.V. as $n \longrightarrow \infty$.

② Apply iid → $M_r(s) = [M_x(s)]^n$

③ Reminder: if $Y = ar + b$
how is $M_y(s)$ related to $M_r(s)$?

$$M_y(s) = E(e^{-sy}) = E[e^{-ars} e^{-bs}]$$

$$= e^{-sb} \int_{-\infty}^{+\infty} e^{-asr_0} f_r(r_0)\, dr_0$$

$$M_y(s) = e^{-sb} M_r(as) = e^{-sb} [M_x(as)]^n$$

$$y = \frac{r - E(r)}{\sigma_r}$$

$$= \frac{r - nE(x)}{\sqrt{n}\,\sigma_x}$$

$$\Rightarrow a = \frac{1}{\sqrt{n}\,\sigma_x}, \quad b = -\frac{\sqrt{n}\,E(x)}{\sigma_x}$$

$$M_y(s) = e^{+s\frac{\sqrt{n}\,E(x)}{\sigma_x}} \left[M_x\left(\frac{s}{\sqrt{n}\,\sigma_x} \right) \right]^n$$

$$M_y(s) = \left[e^{\frac{s\,E(x)}{\sqrt{n}\,\sigma_x}} \; M_x\left(\frac{s}{\sqrt{n}\,\sigma_x} \right) \right]^n$$

↖ Transform of y when

$$y = \frac{x_1 + x_2 + x_3 + \cdots + x_n - n\,E(x)}{\sqrt{n}\,\sigma_x}$$

(4) approx $e^{\cdots}$, M_x near $s \approx 0$

$$e^x \simeq 1 + x + \frac{x^2}{2}$$

$$e^{s\frac{E(x)}{\sigma_x\sqrt{n}}} \simeq 1 + \frac{E(x)}{\sigma_x}\left(\frac{s}{\sqrt{n}}\right) + \frac{[E(x)]^2}{2\sigma_x^2}\left(\frac{s}{\sqrt{n}}\right)^2$$

$$M_x\left(\frac{s}{\sqrt{n}\,\sigma_x}\right)$$

$$M_x(s) = E\left[e^{-sx}\right] \simeq E\left[1 - sx + \frac{s^2x^2}{2}\right]$$

$$= 1 - s\,E(x) + \frac{1}{2}s^2 E[x^2]$$

$$\Rightarrow M_x(s) \simeq 1 - s\,E(x) + \frac{1}{2}s^2\,E[x^2]$$

$$M_x\left(\frac{s}{\sqrt{n}\,\sigma_x}\right) \simeq 1 - \frac{E(x)}{\sigma_x}\left(\frac{s}{\sqrt{n}}\right) + \frac{E(x^2)}{2\sigma_x^2}\left(\frac{s}{\sqrt{n}}\right)^2$$

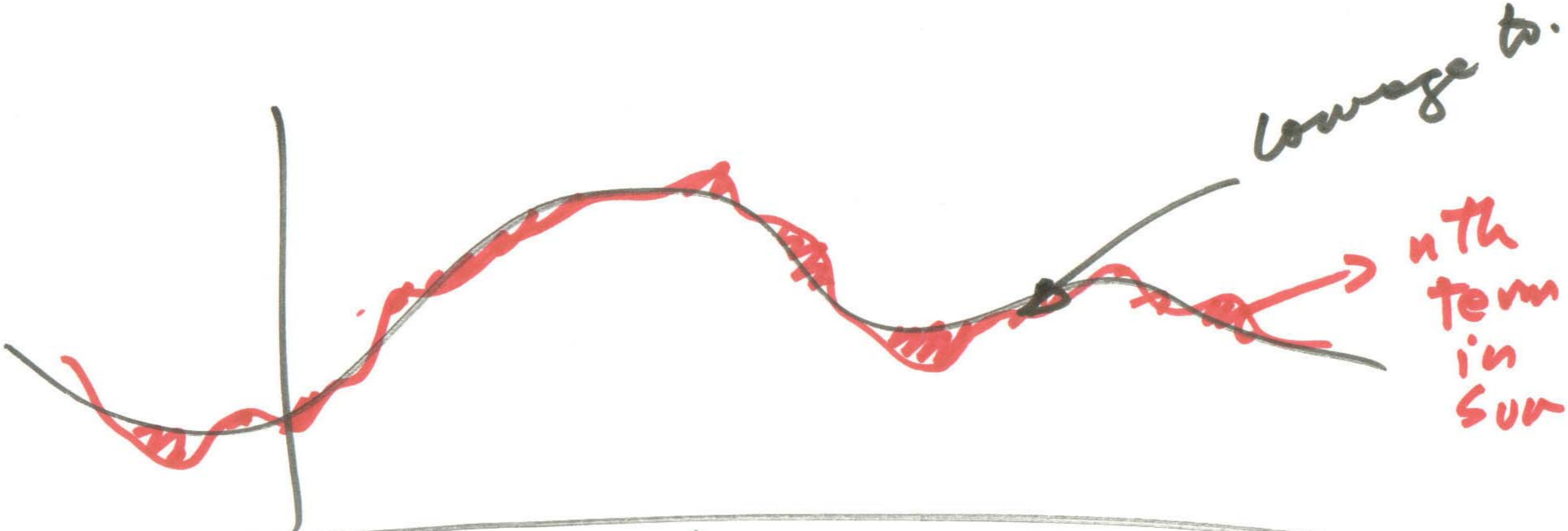
⑤ multiply approx for $e^{\cdots}$ and M_{x}, collect Terms. plug it into $M_y(s) = \left[e^{\cdots} \; M_x(\ldots) \right]^n$

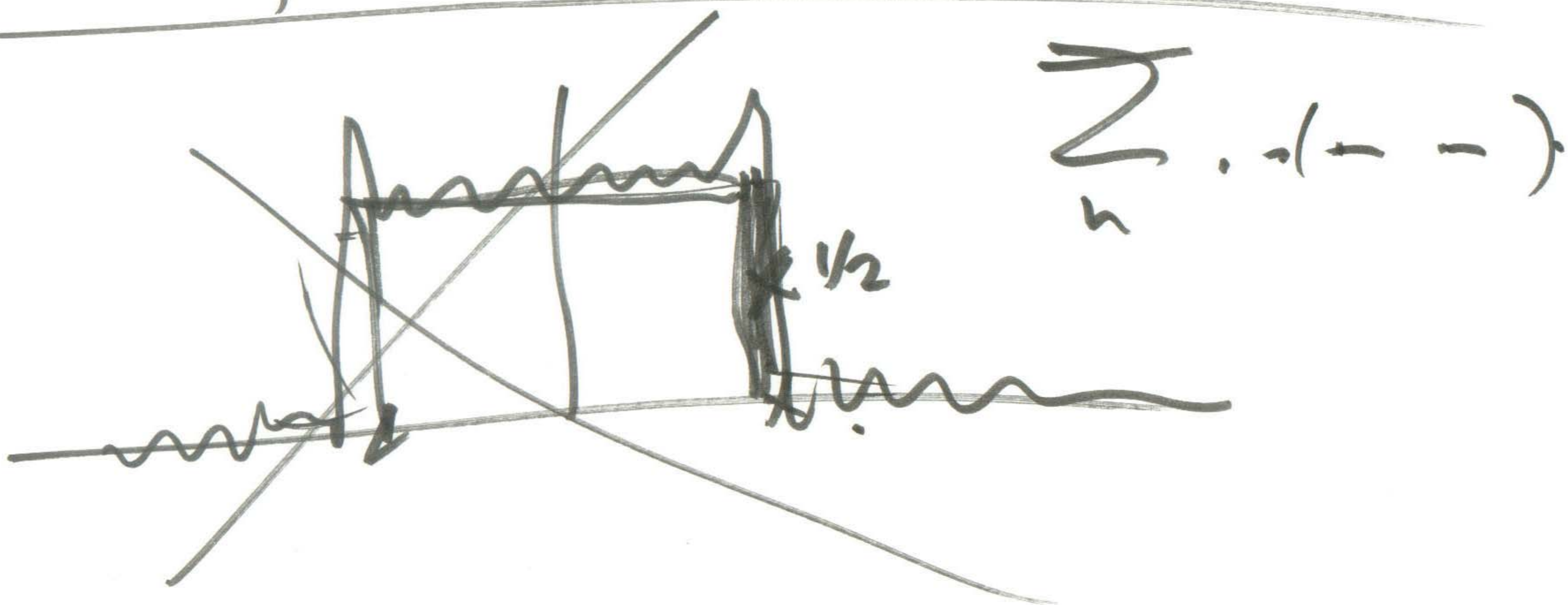
$$M_y(s) = \left(1 + \frac{1}{2}\frac{s^2}{n}\right)^n \qquad s^2/2$$

⑥ take $\lim_{n\to\infty} M_y(s) = e^{s^2/2}$

Trafo of a gaussian mean 0, s.d 1

This does not assume point by point convergence yet!


Converge to.
nth term in sum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(7) Invoke: Continuity thm of Xform theory:

if $\lim_{n \to \infty} M_{y_n}(s) = M_w(s)$ and if

$M_w(s)$ is a continous function Then

CDF of R.V y_n converge to CDF of R.V w, on a point by point basis.

$$\lim_{n \to \infty} F_{y_{\leq}}(y_0) = \Phi(y_0)$$

CLT

$$r = X_1 + X_2 + \cdots + X_n$$

$X_1, X_2 \ldots X_n$ IID.

$E(X)$ σ_x^2.

$$\lim_{n \to \infty} \Pr_{\leq}(r_0) = \Phi\left(\frac{r - E(r)}{\sigma_r}\right)$$

$$E(r) = n\,E(x)$$

$$\sigma_r = \sqrt{n}\,\sigma_x$$

Statistics

- Statistic : infer from data.

Statistician : Build Probabilistic models of reality :

- prediction
- gain ~~into~~ insight into behavior
- decisions.

Question To answer

(1) Based on observed exp data, does this <u>Prob.</u> model seem reasonable?

⟶ Significance Testing

(2) Given data, how to choose among <u>Prob models</u>?

⟹ Hypothesis Testing.

(3) Given postulated model of a phys. system + exp data how to establish the most desirable value of parameters for that prob. model.

⟹ estimation.

(4) how to combine Apriori info with exp. data

⟹ Bayesian Analysis.