Markov Chains

- Bernoulli & Poisson $\longrightarrow$ memoryless
  - future doesnot depend on past
- Markov:
  - Future depends on past.
  - Can be predicted by what happened in the past

Markov: Effect of the past on future summarized by a **state**

- state takes on finite # of values.
- changes according to probabilities that don't ~~change over time~~. depend on the time of change.

Discrete Time Markov Chains

- State changes at certain discrete time instants, indexed by n.
- State $X_n \in S$
- S = finite set of states, called state space.
- Markov chain described by Transition Probabilities P_{ij}
- P_{ij}: whenever state is i, probability p_{ij} next state is j.

$$P_{ij} = P(X_{n+1} = j \mid X_n = i) \qquad i,j \in S$$

- Assume Markov property:

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i) = P_{ij}$$

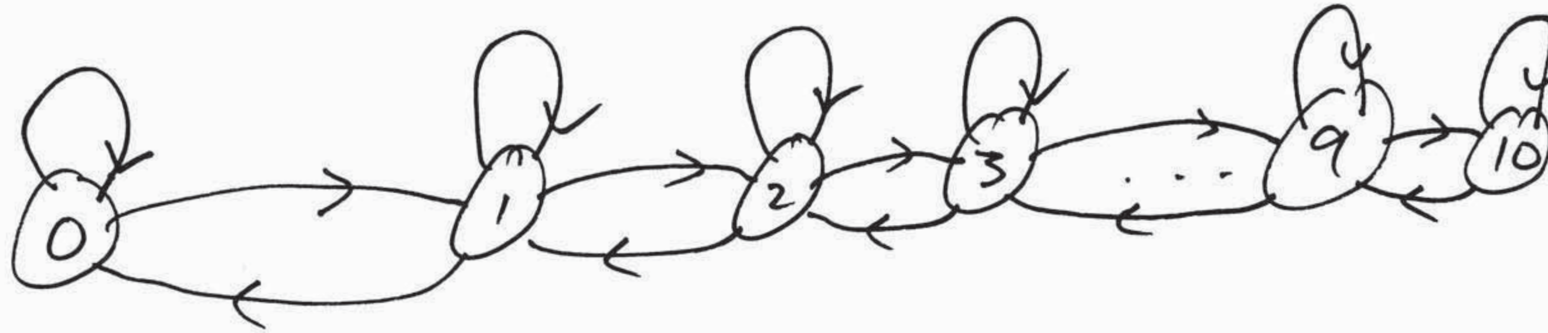
$\forall n$, $\forall$ states $i, j \in S$

and all possible sequences $i_0, \ldots, i_{n-1}$

In Words: - Transition prob. p_{ij} applies whenever state is reached, no matter <u>HOW</u> state i was reached.

- X_{n+1} only depends on past through X_n.

- $$\sum_{j=1}^{m} p_{ij} = 1 \quad \forall i$$



Transition Probability Matrix

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & & & \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

ith row, jth column : p_{ij}

Also known as Transition probability graph.

__Example__: Student either up to date in each week or behind. (2 states).

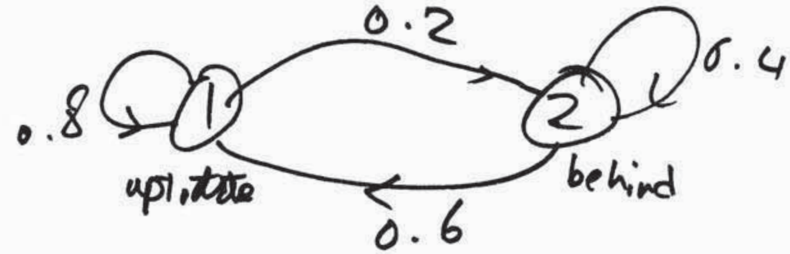
If up to date a given week, then up to date next week with prob 0.8, and behind 0.2.

If behind in a given week, then prob of up to date next week is 0.6, behind is 0.4.

~~$p_{11} = 0.8$ p_{12}~~ State 1 = up to date
State 2 = behind.

$P_{11} = 0.8 \qquad P_{12} = 0.2 \qquad P_{21} = 0.6 \qquad P_{22} = 0.4$

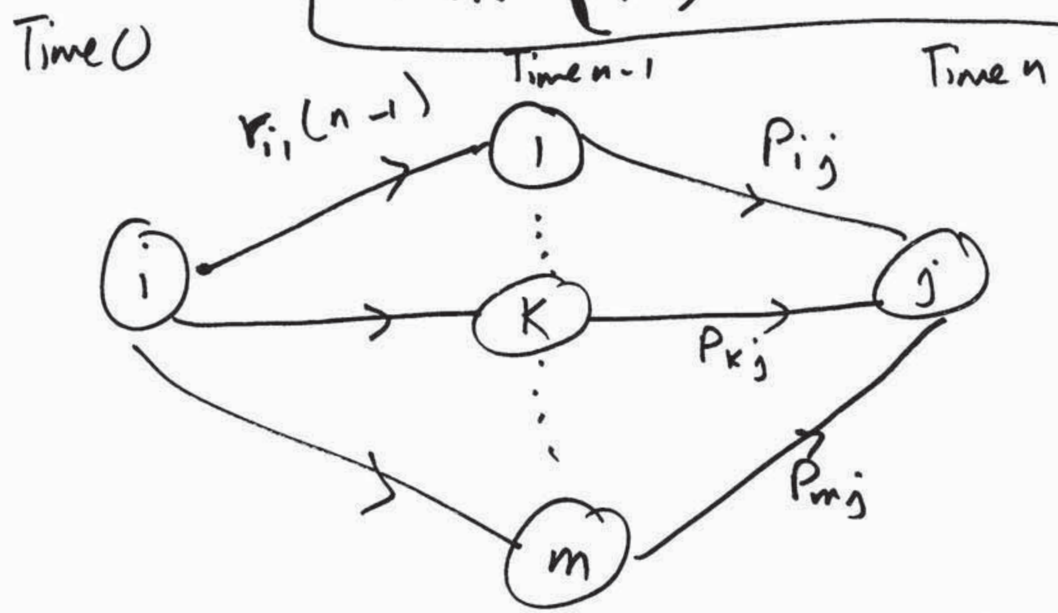
Transition Prob matrix $\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$



~~Probability of rec~~

n-step Transition Probabilities

$$\boxed{r_{ij}(n) = P(X_n = j \mid X_0 = i)} \; ??$$



Apply Chapman-Kolmogorov Eqn:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\, P_{kj} \qquad \forall\, n > 1 \quad \forall\, i, j$$

Need initial Conditions: $r_{ij}(1) = P_{ij}$

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n-Step Transition prob. matrix:

$r_{ij}(n)$ is ith row, jth column.

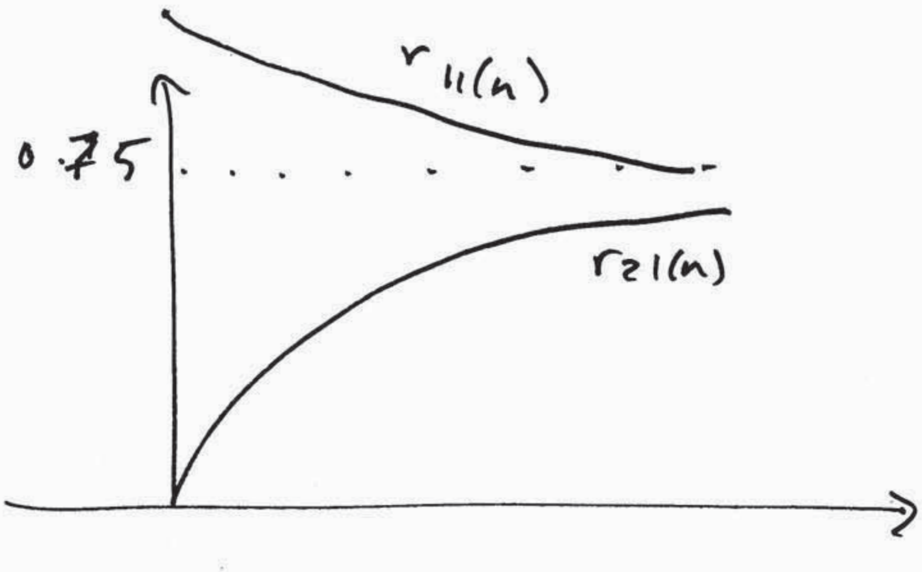
$$\begin{array}{c|cc} & U & B \\ \hline U & 0.8 & 0.2 \\ B & 0.6 & 0.4 \end{array} \qquad \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix} \qquad \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix} \qquad \begin{bmatrix} .75 & .249 \\ .748 & .252 \end{bmatrix} \qquad \begin{bmatrix} .7501 & .2499 \\ .7498 & .2502 \end{bmatrix}$$

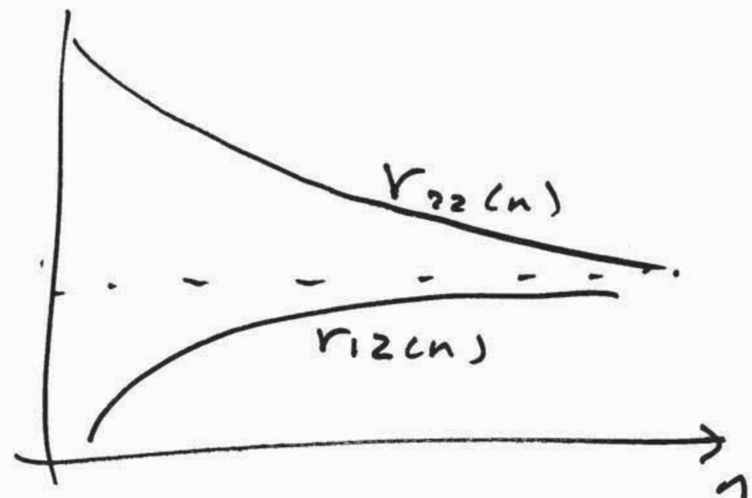
$r_{ij}(1)$ $\quad$ $r_{ij}(2)$ $\quad$ $r_{ij}(3)$ $\quad$ $r_{ij}(4)$ $\quad$ $r_{ij}(5)$

<u>Observe</u>: $r_{ij}(n)$ converges to a limit as $n \to \infty$

Independent of initial state

Conclusion: Each state has a steady state probability of being occupied at time far into the future.





Ex. fly moves along a straight line in unit increments

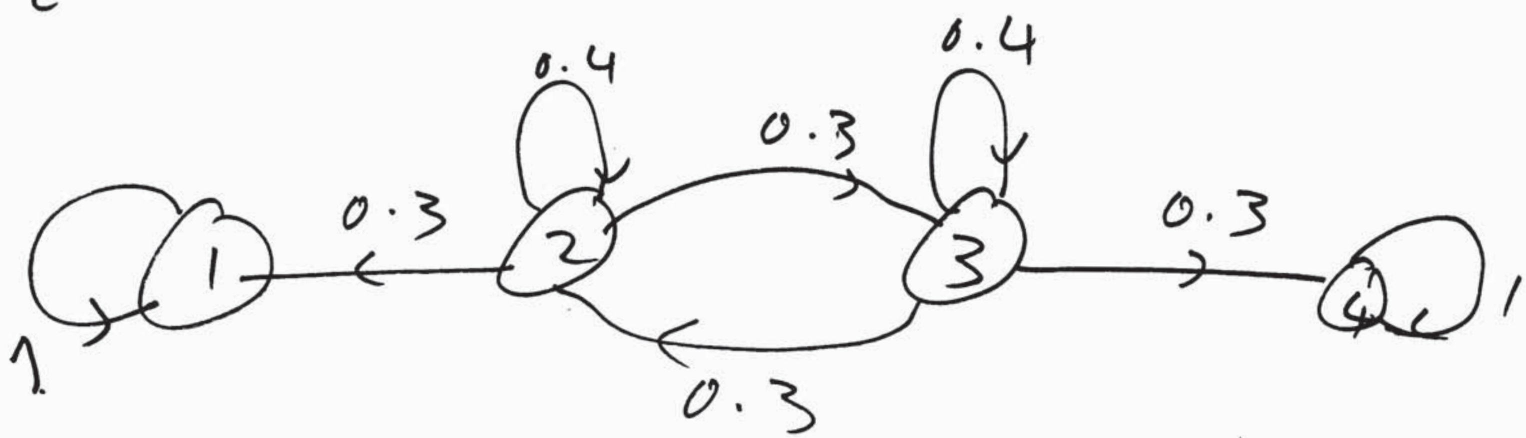
- one unit left with prob 0.3
- " " right " " 0.3
- stays in place " " 0.4
- Two Spiders in position 1 and m

- Fly starts in a position between 1 and m.

- $P_{11} = 1$ $\quad$ $P_{mm} = 1$.

$$P_{ij} = \begin{cases} 0.3 & j = i-1 \text{ or } j = i+1 \\ 0.4 & \text{if } i = j \end{cases}$$

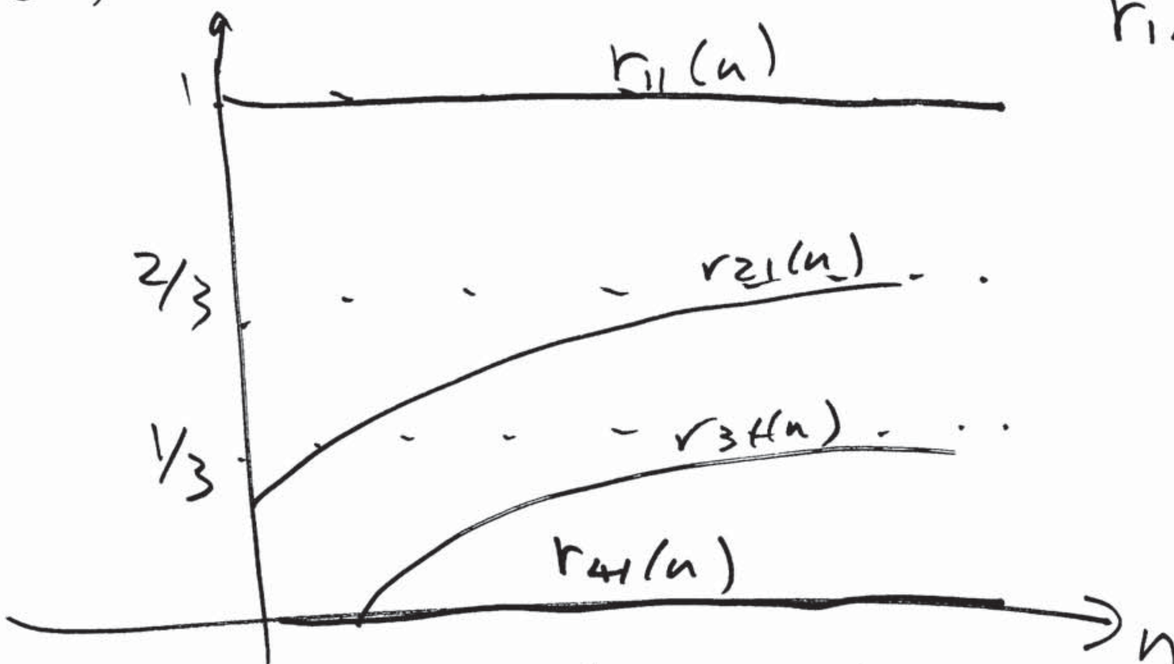
suppose:
$m = 4$



what is $r_{ij}(n)$ as $n \to \infty$

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 0 & 0 & 0 \\
2 & 0.3 & 0.4 & 0.3 & 0 \\
3 & 0 & 0.3 & 0.4 & 0.3 \\
4 & 0 & 0 & 0 & 0
\end{array}
\quad \cdots\cdots \quad
\begin{array}{|cccc|}
\hline
1 & 0 & 0 & 0 \\
2/3 & 0 & 0 & 1/3 \\
1/3 & 0 & 0 & 2/3 \\
0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$r_{ij}(1)$ $\qquad\qquad$ $r_{ij}(\infty)$



prob of reaching state 1 starting from state i

observation: - limit depends on the initial state
$\text{as } n \to \infty$

- "Absorbing" states: once reached ∞ly repeated.

states 1 and 4 absorbing.

- Idea: Classify states based on long term frequency they are visited.

- Define Accessible:

state j is accessible from state i

If $\exists n$ s.th. $r_{ij}(n) > 0$

$A(i)$: set of accessible states from i

- Recurrent & Transient States

Recurrent: state i is recurrent if starting from i, and from wherever you can go There is a way of returning To i.

- **Aliter Def**: i is recurrent if $\forall j$ That is accessible from i, i is also accessible from j.

If not recurrent ⟶ Then **Transient**

- ~~If a recurrent state is visited once, it is going to be visited ∞ # of Times.~~

- **Alitor Def of Transient**
i is Transient if $\exists$ $j \in A(i)$ s.Th. i is not accessible from j.

- ⟹

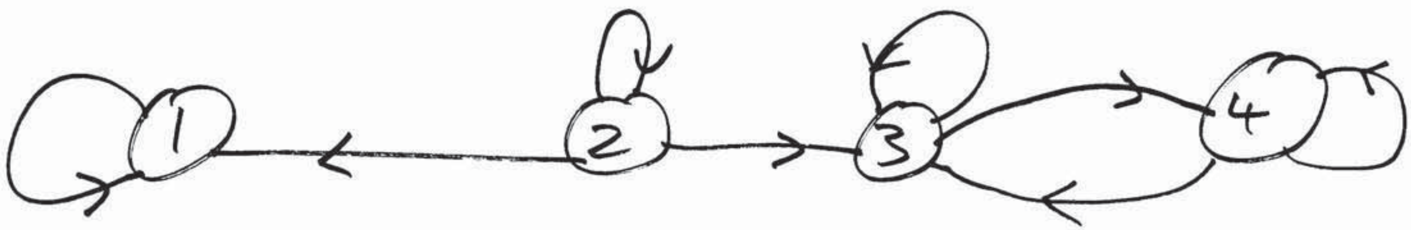
Can show:

① If a recurrent state is visited once, it is going to be visited ∞ly many times.

② A transient state will only be visited finite # of times.

if i transient $\quad P(X_n = i) \to 0$ as $n \to \infty$

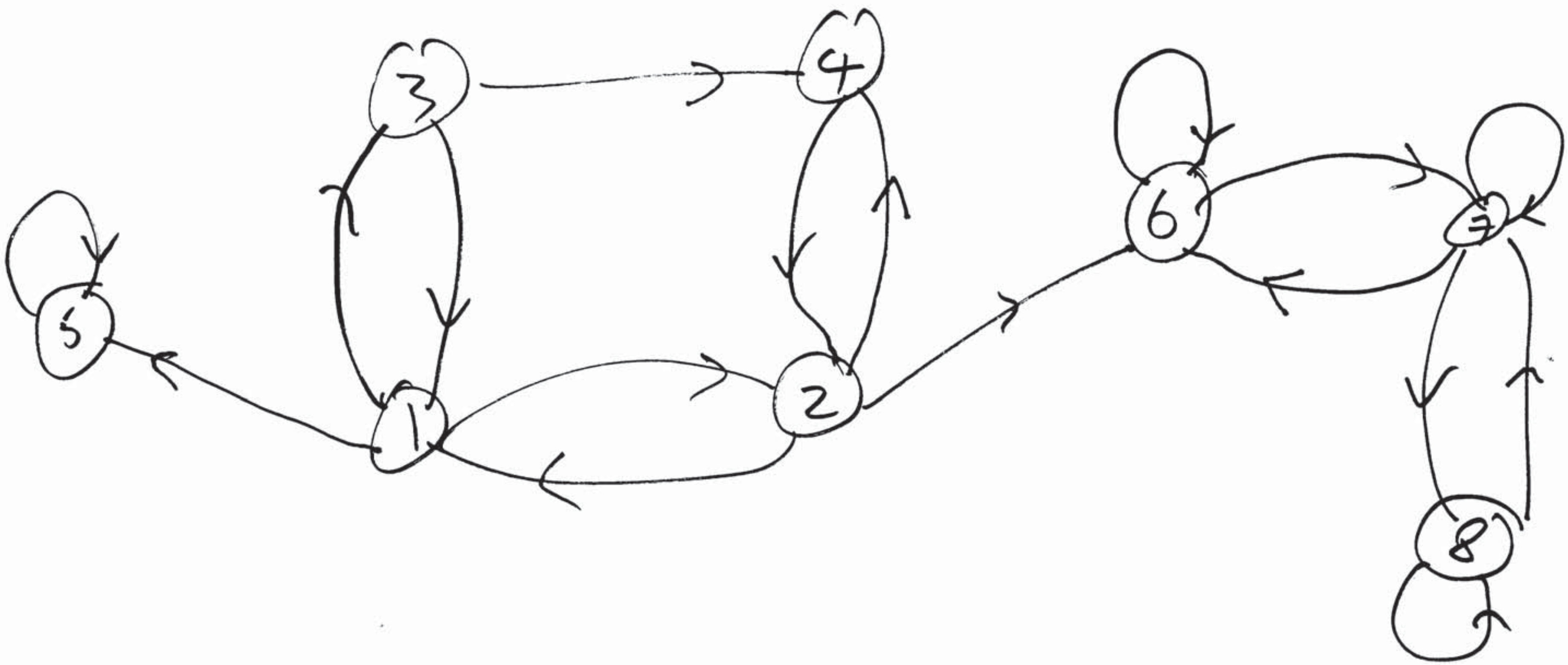
Ex



which states are transient which ones recurrent

1 = recurrent : only 1 is accessible from 1
2 = Transient
3 & 4 : recurrent.

How about This.



Recurrent Class

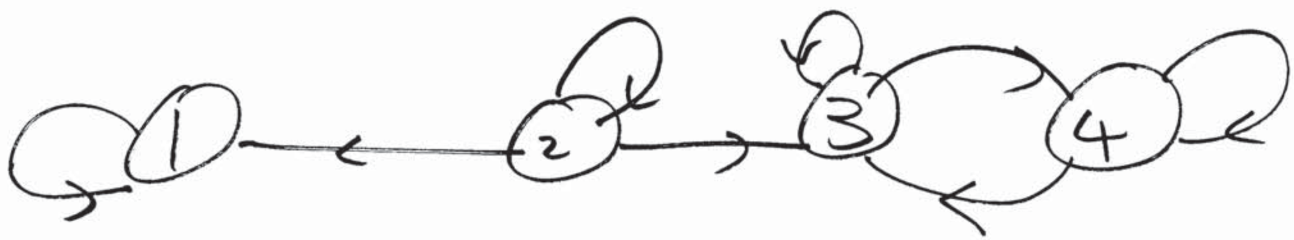
- Collection of recurrent states That communicate with each other and with no other state.

- If i is recurrent, $A(i)$ forms a recurrent class.

  $\Longrightarrow$ states in $A(i)$ are all accessible from each other **And** no state outside of $A(i)$ is accessible from them.

- For a recurrent state i we have $A(i) = A(j) \quad \forall j \in A(i)$

Example



What are recurrent classes?

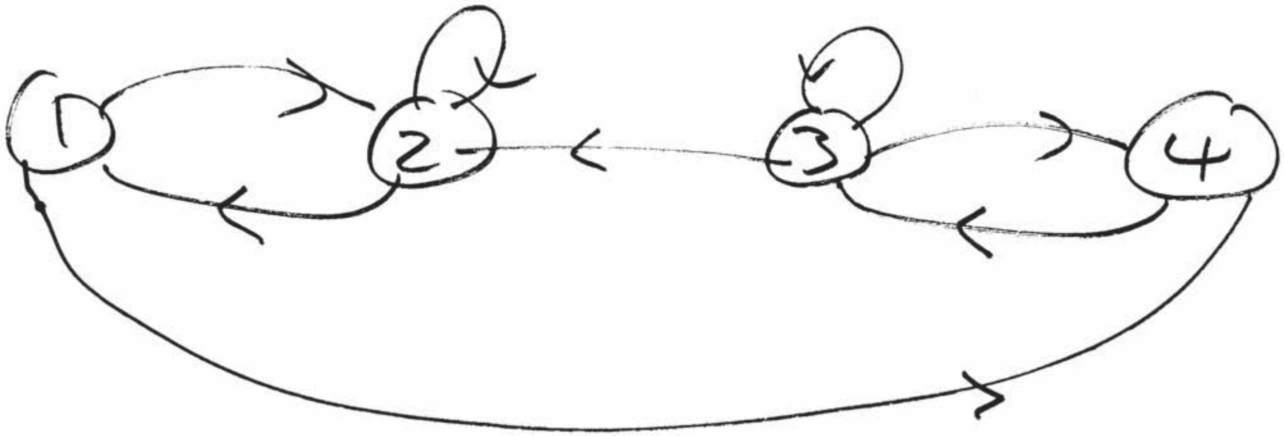
A : ① is recurrent class
3, 4 is " ".

Markov Chain Decompositions

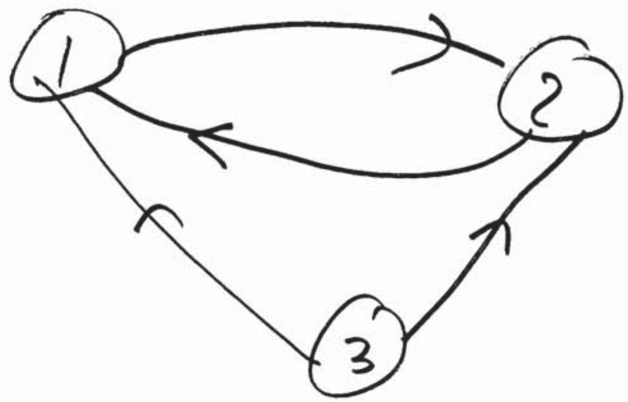
1. A M.C. can be decomposed into 1 or more recurrent classes plus possibly some transient states
2. A recurrent state is accessible from all states in its class but not from recurrent states in other classes.
3. A transient state is not accessible from any recurrent state
4. At least one, maybe more, recurrent states are accessible from a given transient state.

Ex

How many recurrent classes?
how many Transient?



(1)



A: 1 recurrent class: (1 & 2)
1 Transient state (3)

- Two classes of recurrent states.
  - class 1 : state 1
  - " 2 : 4, 5
- Two transient states : 2, 3