Markov Chains

- Bernoulli & Poisson \rightarrow memoryless
  - future does not depend on past

- Markov:
  - Future depends on past
  - Can be predicted by what happened in the past

Markov: Effect of the past on future
- Summarized by a state
- State takes on finite \# of values
- Changes according to probabilities
  - That don't change over time. Depend on the Time of change.
Discrete Time Markov Chains

- State changes at certain discrete time instants.
- Indexed by \( n \).
  - State \( X_n \in S \)
  - \( S \) = finite set of states, called State Space.
- Markov chain described by Transition Probabilities \( P_{ij} \).
- \( P_{ij} \): Whenever state is \( i \), probability \( P_{ij} \) that next state is \( j \).
  \[
P_{ij} = P(X_{n+1} = j \mid X_n = i) \quad i, j \in S
  \]
- Assume Markov property:
  \[
P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i)
  = P_{ij}
  \]
All states $i \in S$ and all possible sequences $i_0, \ldots, i_{n-1}$

In Words: Transition prob. $P_{ij}$ applies whenever state is reached, no matter how state $i$ was reached.

$X_{n+1}$ only depends on past through $X_n$.

$$\sum_{j=1}^{m} P_{ij} = 1 \quad \forall i$$
Transition Probability Matrix

\[
P = \begin{pmatrix}
  p_{11} & p_{12} & \cdots & p_{1n} \\
  p_{21} & p_{22} & \cdots & p_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{m1} & p_{m2} & \cdots & p_{mn}
\end{pmatrix}
\]

\text{ith row, jth column: } p_{ij}

Also known as Transition probability graph.

\text{Example: student either up to date in each week or behind. (2 states).}

If \text{up to date in a given week, then up to date next week with prob 0.8, and behind 0.2.}

If \text{behind in a given week, then prob of up to date next week is 0.6, behind is 0.4.}

\text{State 1 = up to date} \\
\text{State 2 = behind.}
\[ P_{11} = 0.8 \quad P_{12} = 0.2 \quad P_{21} = 0.6 \quad P_{22} = 0.4 \]

Transition Prob matrix:
\[
\begin{bmatrix}
0.8 & 0.2 \\
0.6 & 0.4
\end{bmatrix}
\]

\[
\text{upstate} \quad 0.8 \quad 0.2 \\
0.6 \quad \text{behind} \quad 0.4
\]

\[
\text{Probability of } \mathbf{X}_n
\]

\[
\text{n-Step Transition Probabilities}
\]

\[
\text{Bin}(\mathbf{n}) = P(\mathbf{X}_n = j \mid \mathbf{X}_0 = i)
\]
Apply Chapman-Kolmogorov Eqn:

\[ r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj} \quad \forall n > 1 \]

Need initial conditions: \( r_{ij}(1) = p_{ij} \)

n-step transition prob. matrix:

\( r_{ij}(n) \) is \( i \)'th row, \( j \)'th column.

\[
\begin{array}{cc}
U & B \\
0.8 & 0.2 \\
0.6 & 0.4 \\
\end{array}
\]

\[
\begin{array}{cc}
r_{ij}(1) & \quad \begin{array}{cc}
r_{ij}(2) & \quad \begin{array}{cc}
r_{ij}(3) & \quad \begin{array}{cc}
r_{ij}(4) & \quad \begin{array}{cc}
r_{ij}(5) &
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Observe: \( r_{ij}(n) \) converges to a limit as \( n \to \infty \) independent of initial state.
**Conclusions:** Each state has a steady state probability of being occupied at times far into the future.

![Graphs](image)

**Example:** A fly moves along a straight line in unit increments:
- one unit left with prob 0.3
- one unit right with prob 0.3
- stays in place with prob 0.4

Two spiders, in position 1 and 4.
- Fly starts in a position between $a$ and $m$.

- $P_{11} = 1$
- $P_{ij} = \begin{cases} 0.3 \\ 0.4 \end{cases}$
- $P_{mm} = \frac{1}{2}$
- $\delta = 1 - 1 \quad \text{or} \quad j = i + 1$
- if $i = j$

Suppose:
- $m = 4$

What is $r_{ij}(n)$ as $n \to \infty$
\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 0 & 0 & 0 \\
2 & 0.3 & 0.4 & 0.3 & 0 \\
3 & 0 & 0.3 & 0.4 & 0.3 \\
4 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
2/3 & 0 & 0 & 1/3 \\
1/3 & 0 & 0 & 2/3 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

\[r_{ij}(1)\]

\[r_{ij}(\infty)\]

\[
\begin{array}{c}
\text{prob of reaching state } 1 \text{ starting from state } i \\
\end{array}
\]
Observation: limit depends on the initial state.

"Absorbing" states: once reached, only repeated.
State 1 and 4 absorbing.

Idea: Classify states based on long term frequency they are visited.

Define Accessible:
State j is accessible from state i if for all n, \( r_{ij}(n) > 0 \)

\( A(i) \): set of accessible states from i
Recurrent & Transient States

**Recurrent State:** \( i \) is recurrent if starting from \( i \) and from wherever you can go there is a way of returning to \( i \).

**Alternative Def:** \( i \) is recurrent if \( A(i) \) is accessible from \( i \), \( i \) is also accessible from \( j \).

If not recurrent \( \Rightarrow \) Then Transient

- If a recurrent state is visited once in \( T \) it is \( T \) going to be visited \( \geq T \) times.

**Alternative Def of Transient:** \( i \) is Transient if \( \exists j \in A(i) \) s.t. \( i \) is not accessible from \( j \).
Can show:

1. If a recurrent state is visited once, it is going to be visited only many times.
2. A transient state will only be visited a finite number of times.

If $i$ is transient, $P(X_n = i) \to 0$ as $n \to \infty$

Ex

Which states are transient which ones recurrent?

1 = recurrent: only 1 is accessible from 1
2 = Transient
3 & 4: recurrent.
How about this.
Recurrent Class

- Collection of recurrent states that communicate with each other and with no other state.

- If i is recurrent, A(i) forms a recurrent class.

⇒ States in A(i) are all accessible from each other and no state outside of A(i) is accessible from them.

- For a recurrent state i we have A(i) = A(j) ∀ j ∈ A(i)

Example

What are recurrent classes?
A: 0 is recurrent class
3,4 is not.

Markov Chain Decompositions

1. A Markov Chain can be decomposed into 1 or more recurrent classes plus possibly some transient states.

2. A recurrent state is accessible from all states in its class but is not from recurrent states in other classes.

3. A transient state is not accessible from any recurrent state.

4. At least one, maybe more, recurrent states are accessible from a given transient state.
A: 1 recurrent class: (1 & 2)
1 transient state (3)
- Two classes of recurrent states:
  - Class 1: State 2
    - u 2: 4, 5
- Two transient: 2, 3