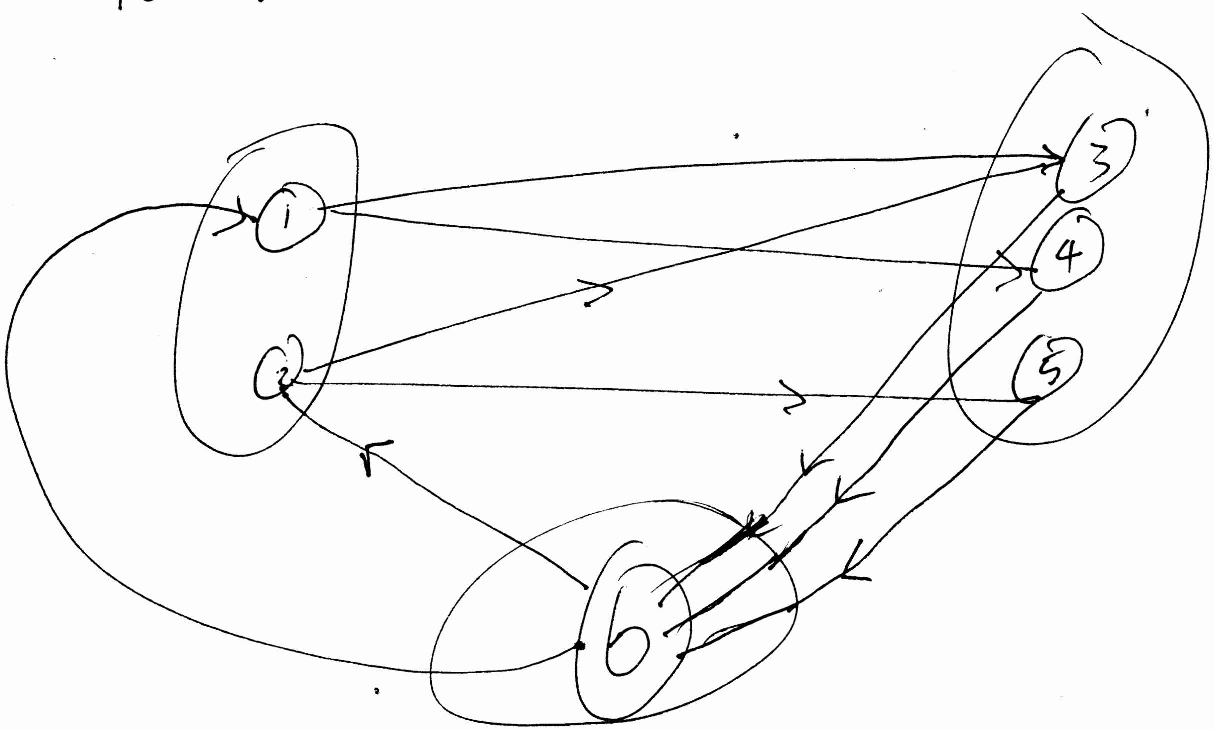
Periodicity

- A recurrent class is periodic if its states can be grouped in $d > 1$ disjoint subsets $S_1, \dots, S_d$. s.th. all transitions from one subset leads to the next subset

Ex:



3 subsets (1, 2)
(3, 4, 5)
(6)

Mathematically:

If $i \in S_k$ and $p_{ij} > 0 \Rightarrow \begin{cases} j \in S_{k+1} & k = 1, \dots, d-1 \\ j \in S_1 & k = d \end{cases}$

Aperiodic reccurent class: not periodic.

Intuitively: in a periodic recurrent class, move through the sequence of subsets in order; after d steps end up in the same subset

Steady State Behavior

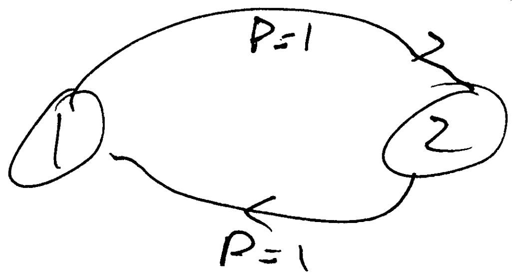
- what happens to $r_{ij}(n)$ as $n \to \infty$
- Does it converge to something?
- If so, is convergence depend on initial condition?

- <u>observation 1</u> : $\lim_{n\to\infty} r_{ij}(n)$ depends on initial state if There are 2 or more recurrent state

<u>why?</u> depends on whether i and j are in The same class!

$\Longrightarrow$ Better focus only on single recurrent class.

<u>Observation 2</u> Consider



Never Converges!

$$r_{ij}(n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

<u>why?</u> It is periodic recurrent class

Can show That for p. r. c. $r_{ij}(n)$ oscillates.

upshot: Assume A single recurrent class that is not periodic

1. Then $\lim_{n\to\infty} r_{ij}(n) = \pi_j \quad \forall i$ i.e. independent of initial state i.

   π_j is called steady state probability for j.

2. π_j are the unique solution to the systems of Equations ~~below~~.

balance equations → $\pi_j = \sum_{k=1}^{m} \pi_k P_{kj} \qquad j = 1, \dots, m$

$$1 = \sum_{k=1}^{m} \pi_k$$

→ can use these to solve for π_j from p_{ij}.

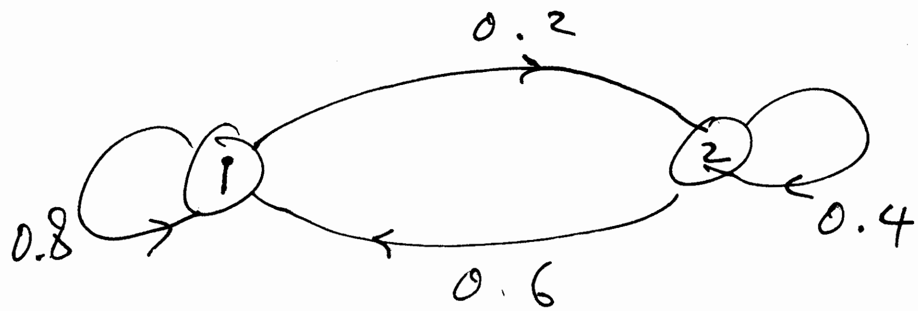
3. $\pi_j = 0$ if j Transient

   $\pi_j > 0$ if j recurrent.

Let $n \to \infty$ in

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) P_{kj} \Rightarrow \pi_j = \sum_{k=1}^{m} \pi_k P_{kj}$$

EX Student up to date / Behind.



$P_{11} = 0.8$
$P_{12} = 0.2$
$P_{21} = 0.6$
$P_{22} = 0.4$

Balance Equations

linearly dependent. $\leftarrow$ $\left\{ \begin{array}{l} \pi_1 = \pi_1 P_{11} + \pi_2 P_{21} \\ \pi_2 = \pi_1 P_{12} + \pi_2 P_{22} \end{array} \right\}$

$$\pi_1 + \pi_2 = 1$$

Typically one of the m Balance Equs can be ~~written~~ derived from the remaining ones.

$\Rightarrow$ 2 Linearly indp. Eqn, 2 unknown $\Rightarrow$

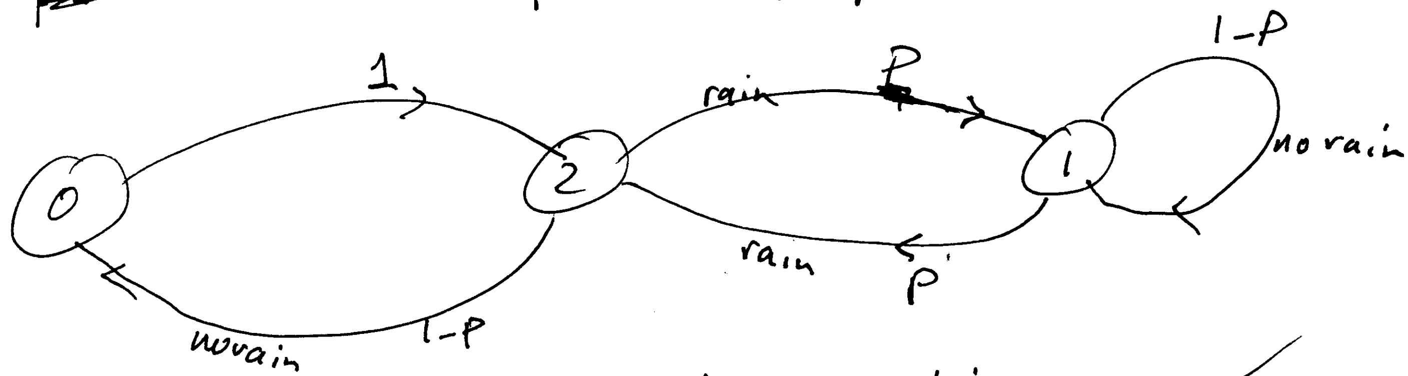
$$\pi_1 = 0.75 \qquad \pi_2 = 0.25$$

EX. Prof commutes from office to home & back
- has 2 umbrellas.
- If it ~~is~~ rains and has an umbrella in her location takes it.
- If ~~raining~~ not raining doesn't take an umbrella

What is the steady state prob. that she gets wet during a commute?

State i : i umbrellas available in her location. $i = 0, 1, 2$.

Draw ~~the~~ Transition prob. graph.



- Single recurrent class, not periodic $\Rightarrow$ ✓

Balance Eqns:

$$\pi_0 = (1-p)\pi_2$$

$$\pi_1 = (1-p)\pi_1 + p\pi_2$$

$$\pi_2 = \pi_0 + p\pi_1$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\Rightarrow \quad \pi_0 = \frac{1-p}{3-p} \qquad \pi_1 = \frac{1}{3-p} \qquad \pi_2 = \frac{1}{3-p}$$

π_0 = steady state prob for being without umbrella

Prob getting wet = $\pi_0 \cdot p(\text{rain}) = \pi_0 p$

Long Term Freq. Interpretation:

- Think of probability as relative frequency in ∞ # of Trials
- EX - machine at the end of each day either OK or broken
  - Each repair cost \$2.
  - Expected cost of repair per day??

Approach ① pick a radom day. find expected repair cost for that day.

$$= \pi_{\text{Broken}} = \text{steady state prob of broken}$$

Approach ② $\dfrac{\text{Total expected repair cost in } n \text{ days}}{n} \quad n \to \infty$

Should get same answer either way.

Let $v_{ij}(n)$ = expected # of visits to j within n steps, starting from i

Assume M.C. is single class aperiodic.

Then

$$\Rightarrow \quad \pi_j = \lim_{n \to \infty} \frac{v_{ij}(n)}{n}$$

π_j = fraction of time in state j

P_{jk} = prob of going from j to k.

$\Rightarrow \pi_j P_{jk}$ = fraction of transitions from j to k.

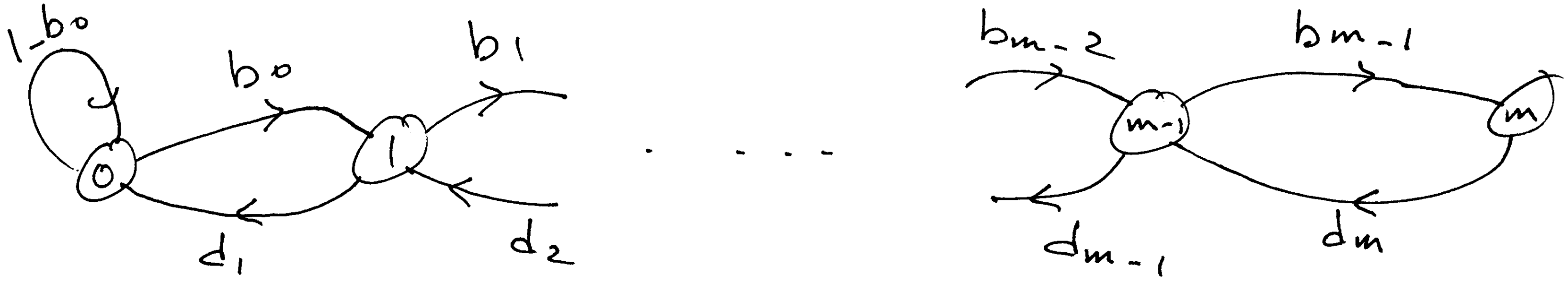
Let $q_{jk}(n)$ = expected # of transitions from j to k over n steps

Then

$$\lim_{n \to \infty} \frac{q_{jk}(n)}{n} = \pi_j P_{jk}$$

Birth_Death Processes

- M.C. ~~[illegible]~~ with linearly arranged states
  Transitions only to neighboring states or self.



b_i = birth prob at ~~[illegible]~~ state i = $P(X_{n+1} = i+1 \mid X_n = i)$

d_i = death " " " " = $P(X_{n+1} = i-1 \mid X_n = i)$

Observe: Transition from $i \to i+1$ must be followed by $i+1 \to i$ before another trans. $i \to i+1$

$\Longrightarrow$ Expected freq. of transitions $i \to i+1$ =
Expected " " " $i+1 \to i$

$$\Longrightarrow \quad \pi_i b_i = \pi_{i+1} d_{i+1} \qquad i = 0, 1, \ldots m-1 \qquad (*)$$

Can show

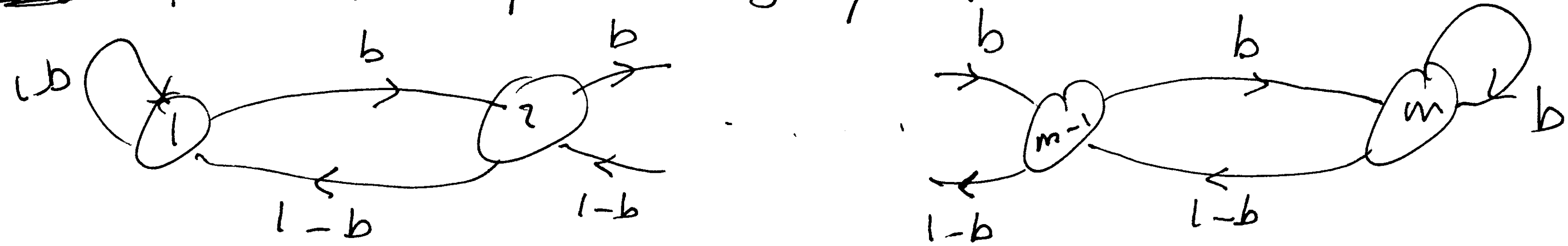
$$\begin{cases} \pi_i = \pi_0 \dfrac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i} & i = 1, \ldots, m \\ \sum_i \pi_i = 1 \end{cases}$$

$\longrightarrow$ Compute all π_i.

Ex

- person walks along a straight line.
- step to right prob b
- " " left prob $1-b$
- states in one of $1, 2, \ldots, m$
- If in state 1 stays in 1 with prob $1-b$
- " " " m " " m " " b

Transition prob. graph:



Find steady state prob Π_i

Apply (*) $\quad \Pi_i b = \Pi_{i+1}(1-b) \Rightarrow \Pi_{i+1} = \rho \Pi_i$

$$\rho = \frac{b}{1-b}$$

$$\Rightarrow \Pi_i = \rho^{i-1} \Pi_1 \qquad i = 1, \dots, m$$

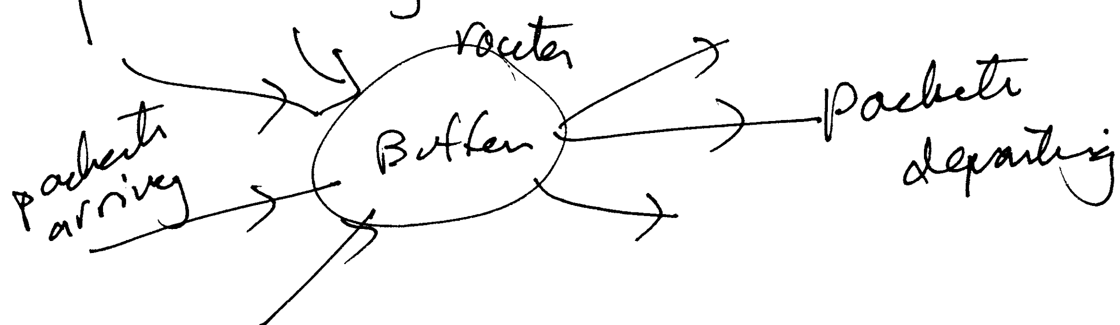
Apply normalization $\quad 1 = \Pi_1 + \dots + \Pi_m \Rightarrow$

$$1 = \Pi_1 (1 + \rho + \rho^2 + \dots \rho^{m-1})$$

$$\Rightarrow \Pi_1 = \frac{1}{1 + \rho + \rho^2 + \dots + \rho^{m-1}}$$

$$\Rightarrow \Pi_i = \frac{\rho^{i-1}}{\sum_{i=0}^{m-1} \rho^i}$$

Ex Queuing theory:



- Buffer capacity m
- If buffer full (has m packets), new packets dropped
  - Discretize Time into small periods.
  - Each period, one of 3 things can happen.

    (1) new packet arrives w/ prob $b > 0$

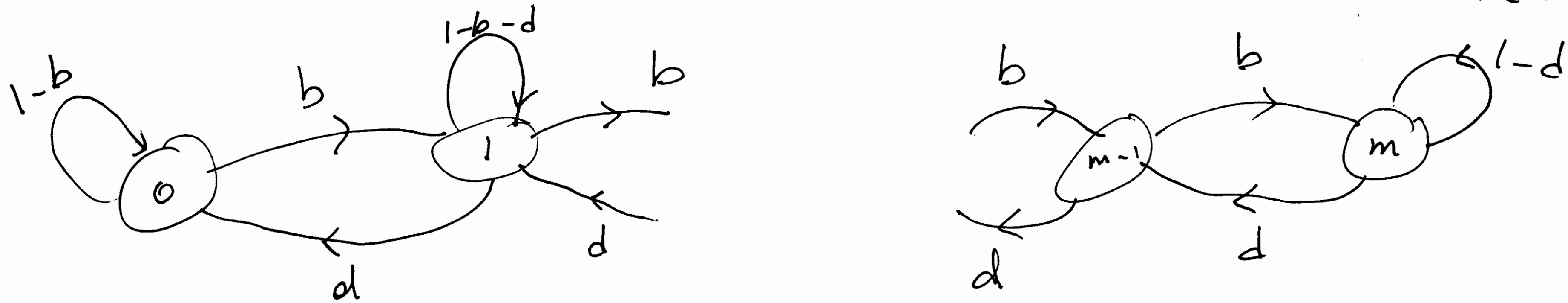
    (2) an existing packet leaves w/ prob $d > 0$ (if There is ~~a packet~~ at least one packet)

    (3) ~~not~~ neither 1 nor 2.

    prob $1-b-d$ if at least one packet

    prob $1-b$ otherwise

Draw Transition prob graph. States $0, 1, \ldots m$ = # of packets in The buffer



Compute steady state probabilities

$(*) \Rightarrow \quad \pi_i \, b = \pi_{i+1} \, d \qquad i = 0, 1, \dots m-1$

Define $\quad \rho = \dfrac{b}{d} \Rightarrow \pi_{i+1} = \rho \, \pi_i$

$$\left.\begin{array}{ll} \pi_i = \rho^i \, \pi_0 & i = 0, 1, \dots m \\ \text{Normalization} \quad 1 = \pi_0 + \pi_1 + \dots + \pi_m & \end{array}\right\} \Rightarrow$$

$$1 = \pi_0 \left(1 + \rho + \dots + \rho^m\right)$$

$$\Rightarrow \quad \pi_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{m+1}} & \rho \neq 1 \\[2ex] \dfrac{1}{m+1} & \rho = 1 \end{cases}$$

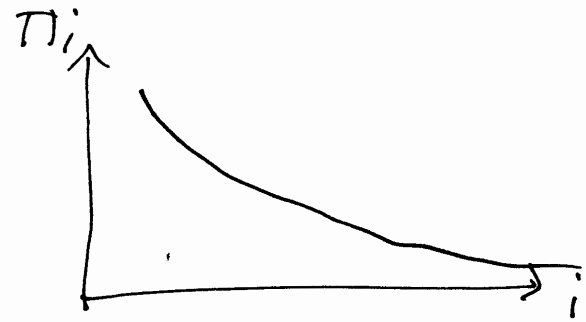
$$\pi_i = \begin{cases} \dfrac{1-\rho}{1-\rho^{m+1}} \rho^i & \rho \neq 1 \\[2ex] \dfrac{1}{m+1} & \rho = 1 \end{cases}$$

what happens as buffer size $m \to \infty$?

Two cases.

(1) $b < d$ i.e $\rho < 1$ i.e more departures than arrivals

$1 - \rho^{m+1} \to 1$ $\pi_i \to \rho^i (1-\rho)$ $\forall i$



$\Rightarrow$ # of packets in the buffer doesnot grow. because $b < d$

(2) $b > d$ i.e. $\rho > 1$ $\Rightarrow$ more arrivals than departure

as $m \to \infty$ $\pi_i \to 0$ $\forall i$ $\Rightarrow$ all states transient $\Rightarrow$ no steady state distribution.

π_i increases with i

$\Rightarrow$ # of packets in the buffer increases

Absorption properties

<u>Def</u>: Recurrent state k is absorbing if

$$P_{kk} = 1 \qquad P_{kj} = 0 \qquad \forall j \neq k$$

- Suppose M.C. starts at a transient state.
  - What is the first recurrent state it enters?
  - How long does it take for this to happen?

- <u>Observe</u>: Answer is same regardless of whether we use first recurrent or first absorbing.

<u>Why?</u> Don't care what happens after recurrent state is reached

$\Rightarrow$ Focus on the case where every recurrent state is absorbing.

Absorption Probability Equs

- Consider a M.C each state either Transient or absorbing
- Fix an absorbing state s
- a_i = prob eventually reaching s starting from i

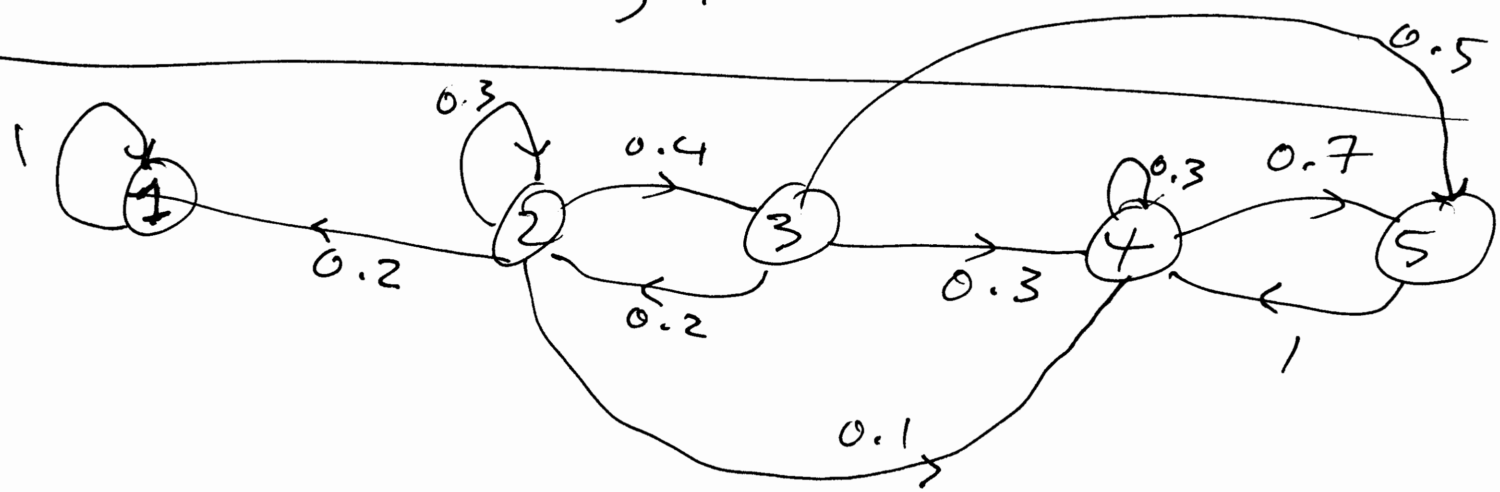
Then: can find a_i by solving ~~These eqns:~~

$$a_s = 1$$

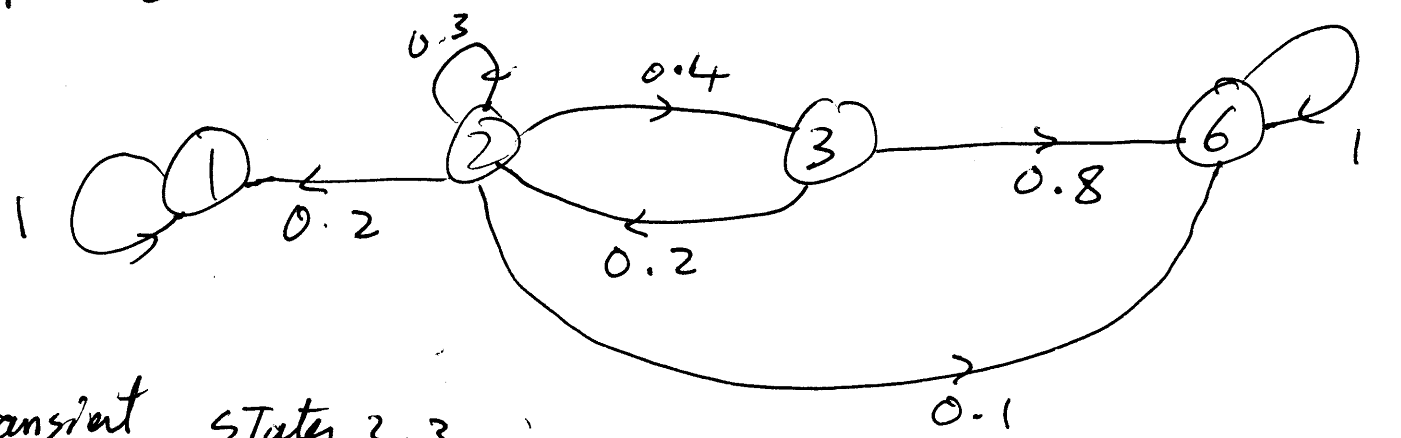
$$a_i = 0 \quad \forall \text{ absorbing } i \neq s$$

$$a_i = \sum_{j=1}^{m} p_{ij} a_j \quad \forall \text{ Transient } i$$

Ex



- Find prob state reaches recurrent class $\{4,5\}$ starting from Transient state
- lump $\{4,5\}$ into $\{6\}$.



For Transient states 2, 3 :

$$\left\{\begin{aligned} a_2 &= 0.2\, a_1 + 0.3\, a_2 + 0.4\, a_3 + 0.1\, a_6 \\ a_3 &= 0.2\, a_2 + 0.8\, a_6 \end{aligned}\right\} \Rightarrow$$

Use $a_1 = 0$ $\quad a_6 = 1$

$$\left.\begin{aligned} a_2 &= 0.3\, a_2 + 0.4\, a_3 + 0.1 \\ a_3 &= 0.2\, a_2 + 0.8 \end{aligned}\right\} \Rightarrow$$

$$a_2 = \frac{21}{31} \qquad a_3 = \frac{29}{31}$$