Periodicity

A recurrent class is periodic if its states can be grouped in \( d \geq 1 \) disjoint subsets \( S_1, \ldots, S_d \). s.th. all transitions from one subset leads to the next subset.

Ex:

\[
3 \text{ subsets: } (1, 2), (3, 4, 5), (6)
\]
Mathematically:

$$\text{If } i \in S_k \text{ and } pij > 0 \Rightarrow \begin{cases} j \in S_{k+1} & \text{for } k = 1, \ldots, d-1 \\ j \in S_d & \text{for } k = d \end{cases}$$

Aperiodic recurrent class: not periodic.

Intuitively: in a periodic recurrent class, move through the sequence of subsets in order; after d steps end up in the same subset.

Steady State Behavior

- What happens to $rij(n)$ as $n \to \infty$?
- Does it converge to something?
- If so, is convergence depend on initial condition?
Observation 1: \( \lim_{n \to \infty} \text{Rij}(n) \) depends on initial state if there are 2 or more recurrent state.

Why? It depends on whether \( i \) and \( j \) are in the same class!

\[ \Rightarrow \text{Better focus only on single recurrent class.} \]

Observation 2: Consider

\[ \text{P=1} \quad 1 \quad 2 \]
\[ \text{P=1} \]

\[ \text{Never Converges!} \]

\[ \text{rij}(n) = \begin{cases} 1 & \text{even} \\ 0 & \text{odd} \end{cases} \]

Why? It is periodic recurrent class.

Can show that for P. R. C. \( \text{Rij}(n) \) oscillates.
Assume a single recurrent class that is not periodic.

1. \( \lim_{n \to \infty} R_{ij}(n) = \Pi_j \) if i.e. independent of initial state i.

\( \Pi_j \) is called steady state probability for j.

2. \( \Pi_j \) are the unique solution to the systems of equations balance equations:

\[ \Pi_j = \sum_{k=1}^{m} \Pi_k P_{kj} \quad j = 1, \ldots, m \]

\[ 1 = \sum_{k=1}^{m} \Pi_k \quad \Rightarrow \text{Can use these to solve for } \Pi_j \text{ from } \Pi_{ij}. \]

3. \( \Pi_j = 0 \) if j transient.

\( \Pi_j > 0 \) if j recurrent.

\[ \lim_{n \to \infty} R_{ij}(n) = \sum_{k=1}^{m} R_{ik}(n-1) P_{kj} \quad \Rightarrow \quad \Pi_j = \sum_{k=1}^{m} \Pi_k P_{kj} \]
EX. Student up to date/Behind.

\[ \begin{align*}
0.2 & \\
0.8 & \\
0.6 & \\
0.4 & \\
\end{align*} \]

Balance Equation:

\[
\begin{cases}
P_1 = P_1, P_{11} + P_{12} P_{21} \\
P_2 = P_1, P_{12} + P_{22} P_{22}
\end{cases}
\]

\[ N_1 + N_2 = 1 \]

Typically, one of the m Balance Equations can be derived from the remaining ones.

\[ \Rightarrow \text{2 Linearly independent Equns, 2 unknowns} \]

\[ \begin{align*}
N_1 &= 0.75 \\
N_2 &= 0.25
\end{align*} \]

EX. Prof. commutes from office to home & back.

- has 2 umbrellas.
- if it rains and has an umbrella in her location, takes it.
- if not raining doesn't take an umbrella.
What is the steady state prob. that she gets wet during a commute?

State $i$: $i$ umbrellas available in her location. $i = 0, 1, 2$

Draw transition prob. graph.

- Single recurrent class, not periodic.

Balance Eqns:

$\pi_0 = (1-p) \pi_2$

$\pi_1 = (1-p) \pi_1 + p \pi_2$

$\pi_2 = \pi_0 + p \pi_1$

$\pi_0 + \pi_1 + \pi_2 = 1$

$\Rightarrow \pi_0 = \frac{1-p}{3-p}$

$\pi_1 = \frac{1}{3-p}$

$\pi_2 = \frac{1}{3-p}$
\[ P_0 = \text{steady state prob for being without umbrella} \]
\[ P_\text{getting wet} = P_0 \cdot p(\text{rain}) = P_0 p \]

**Long Term Freq. Interpretation:**

- Think of probability as relative frequency in \( \omega \) # of trials

**EX:** machine at the end of each day either OK or broken
- Each repair cost $2.
- Expected cost of repair per day??

Approach 1

1. Pick a random day. Find expected repair cost for that day.
2. \[ P_{\text{broken}} = \text{steady state prob of broken} \]

Approach 2

\[ \text{Total expected repair cost in } n \text{ days} \]
\[ \frac{\text{Total expected repair cost in } n \text{ days}}{n} \]

Should get same answer either way.
Let $\text{VI}_j(n) = \text{expected \# of visits to } j \text{ within } n \text{ steps, starting from } i$.

Assume M.C. is single class a periodic.

Then

$$\Rightarrow \text{ lim } \frac{\text{VI}_j(n)}{n} = \Pi_j$$

$\Pi_j$ = fraction of time in state $j$.

$P_{jk} = \text{prob of going from } j \text{ to } k$.

$\Rightarrow \Pi_j \cdot P_{jk} = \text{fraction of Transitions from } j \text{ to } k$.

Let $\text{VI}_j(k) = \text{expected \# of Transitions from } j \text{ to } k \text{ over } n \text{ steps}$.

Then

$$\Rightarrow \text{ lim } \frac{\text{VI}_j(k)}{n} = \Pi_j \cdot P_{jk}$$
Birth-Death Processes

- M.C. in states with linearly arranged states, transitions only to neighboring states or self.

\[ b_i = \text{birth prob at state } i = P(X_{n+1} = i + 1 \mid X_n = i) \]
\[ d_i = \text{death prob at state } i = P(X_{n+1} = i - 1 \mid X_n = i) \]

Observe: Transition from \( i \to i+1 \) must be followed by \( i+1 \to i \) before another trans. \( i \to i+1 \)

\[ \Rightarrow \text{Expected freq. of transitions } i \to i+1 = \]
\[ \text{Expected } \prod_{i \to i+1} \]
\[ \Rightarrow \prod_{i=0,1,...,m-1} b_{i+1} d_i \]
Can show \[ \Pi_i = \Pi_0 \frac{b_0 b_1 \ldots b_{i-1}}{d_1 d_2 \ldots d_i} \quad i = 1, \ldots, m \]

\[ \sum_{i=1}^{m} \Pi_i = 1 \]

Compute all \( \Pi_i \).

**Example**
- Person walks along a straight line.
- Step to right prob \( b \)
- Step to left prob \( 1 - b \)
- Starts in one of \( 1, 2, \ldots, m \)
- If in state 1 stays in 1 with prob \( 1 - b \)
- \( a \) \( m \) \( a \) \( m \) \( a \) \( m \) \( a \) \( m \) \( a \)

**Transition probability graph:**

- Transition graph

\( \begin{array}{c}
\text{Transition prob. graph:}
\end{array} \)
Find steady state prob $H_i$:

Apply $\phi$

$H_i ; b = H_{i+1} (1-b) \Rightarrow H_{i+1} = \rho H_i$

$\rho = \frac{b}{1-b}$

$\Rightarrow H_i = \rho^{i-1} H_1, \quad i = 1, \ldots, m$

Apply normalization

$1 = H_1 + \ldots + H_m \Rightarrow$

$1 = H_1 (1 + \rho + \rho^2 + \ldots + \rho^{m-1})$

$\Rightarrow H_i = \frac{1}{1 + \rho + \rho^2 + \ldots + \rho^{m-1}}$

$\Rightarrow H_i = \frac{\rho^{i-1}}{\sum_{i=0}^{m-1} \rho^i}$

Ex Query Theory:

Buffer

$\begin{array}{c}
\text{packets arriving} \\
\text{packets departing}
\end{array}$
- Buffer capacity \( m \)
- If buffer full (has \( m \) packets), new packets dropped
- Discretize time into small periods.
  - Each period, one of 3 things can happen.
    1. new packet arrives w/ prob \( b > 0 \)
    2. an existing packet leaves w/ prob \( d > 0 \)
       (if there is at least one packet)
    3. neither 1 nor 2.
       prob \( 1 - b - d \) if at least one packet
       prob \( 1 - b \) otherwise.

Draw transition prob graph. States 0, 1, \ldots, \( m \) = # packets in the buffer.

Compute steady state probabilities.
Define \( \rho = \frac{b}{d} \) \( \Rightarrow \Pi_{i+1} = \rho \Pi_i \).

\[ \Pi_i = \rho^i \Pi_0 \quad i = 0, 1, \ldots, m \]

Normalized:

\[ 1 = \Pi_0 + \rho \Pi_1 + \ldots + \rho^m \]

\[ l = \Pi_0 + \rho \left( \frac{1 - \rho}{1 - \rho^{m+1}} \right) + \rho^m \left( \frac{1}{m+1} \right) \]

\[ \Pi_0 = \begin{cases} \frac{l - \rho}{1 - \rho^{m+1}} & \rho \neq 1 \\ \frac{1}{m+1} & \rho = 1 \end{cases} \]

\[ \Pi_i = \begin{cases} \frac{l - \rho}{1 - \rho^{m+1}} \rho^i & \rho \neq 1 \\ \frac{1}{m+1} & \rho = 1 \end{cases} \]
what happens as buffer size $m \to \infty$?

Two cases.

1. $b < d$ i.e. $\rho < 1$ i.e. more departures than arrivals.
   
   $$1 - \rho^{m+1} \to 1$$
   
   $\Rightarrow$ # of packets in the buffer does not grow.
   
   because $b < d$

2. $b > d$ i.e. $\rho > 1$ $\Rightarrow$ more arrivals than departures
   
   as $m \to \infty$ $N_i \to 0$ $\forall i \Rightarrow$ all states transient
   
   $N_i$ increase $\Rightarrow$ # of packet in the buffer increases.

$\Rightarrow$ # of packets in the buffer increases.
Absorption properties

Def. Recurrent state is absorbing if

\[ P_{k,k} = 1 \quad P_{k,j} = 0 \quad \forall j \neq k \]

- Suppose M.C. starts at a transient state.
  - What is the first recurrent state it enters?
  - How long does it take for this to happen?

- Observe: Answer is same regardless of whether we use first recurrent or first absorbing.

Why?

\[ \Rightarrow \text{Focus on the case where every recurrent state is absorbing.} \]
Absorption Probability Equations

Consider a Markov chain with states either transient or absorbing.

- Fix an absorbing state $s$.

- $a_i = \text{prob. eventually reaching } s \text{ starting from } i$.

Then, we can find $a_i$ by solving the equations:

$$a_s = 1$$

$$a_i = 0 \quad \forall \text{ absorbing } i \neq s$$

$$a_i = \sum_{j=1}^{m} p_{ij} a_j \quad \forall \text{ transient } i$$

Ex.

[Diagram of Markov chain with states and transition probabilities labeled.]
- Find prob state reaches recurrent class $\xi_{4,5,3}$ starting from transient state $\xi_{4,5,3}$ into $\xi_{6,3}$.

For transient state $2,3$:

\[
\begin{align*}
\begin{cases}
a_2 &= 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6 \\
a_3 &= 0.2a_2 + 0.8a_6
\end{cases}
\end{align*}
\]

Use $a_1 = 0$, $a_6 = 1$

\[
\begin{align*}
a_2 &= 0.3a_2 + 0.4a_3 + 0.1a_6 \\
a_3 &= 0.2a_2 + 0.8 \\
a_2 &= \frac{21}{31} \\
a_3 &= \frac{29}{31}
\end{align*}
\]