Expected Time To Absorption

- Expected # of steps until a recurrent state is entered.
  i.e. - Time To Absorption

- $\forall$ state i :

$$\mu_i = E[\# \text{ of transitions until absorption, starting from } i]$$
$$= E[\min\{n \geq 0 \mid X_n \text{ is recurrent}\} \mid X_0 = i]$$

Expected Time To Absorption $\mu_1, \mu_2, \ldots, \mu_m$ are The unique soln To The equations :

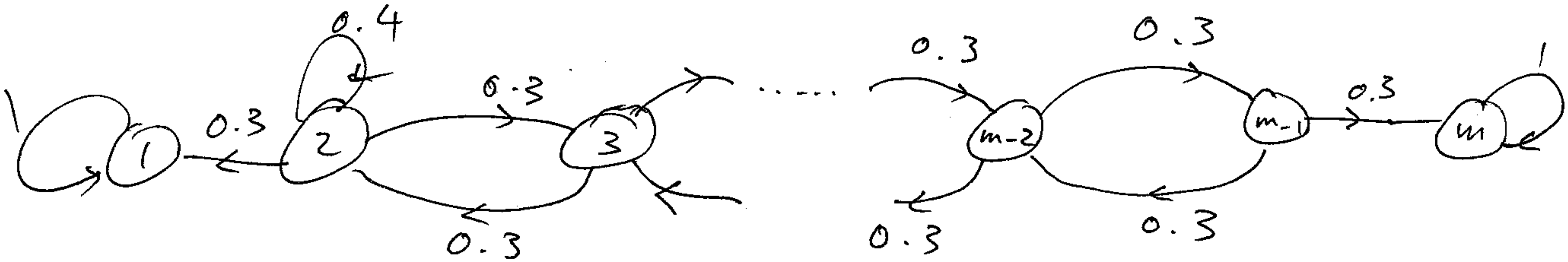
$$\mu_i = 0 \qquad \forall \text{ recurrent state } i$$

$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij} \mu_j \qquad \forall \text{ Transient states } i$$

(*) From Total Expectation Thm.

Words: Time To absorption from Transient state i is 1 plus Time To absorption from The next state j with probability p_{ij}

EX Spider + Fly example:



- Which are absorbing states?

- $1, m \Rightarrow \mu_1 = \mu_m = 0$

Apply (*) $\mu_i = 1 + 0.3\mu_{i-1} + 0.4\mu_i + 0.3\mu_{i+1}$

$i = 2, \ldots, m-1$

Solve by successive substitution:

EX $m=4$

$$\left.\begin{aligned} &\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3 \Rightarrow \mu_2 = \frac{1}{0.6} + \frac{1}{2}\mu_3 \\ &\mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \end{aligned}\right\} \Rightarrow$$

$$\mu_3 = \frac{16}{3} \Rightarrow \mu_2 = 10/3$$

Mean First Passage Time & Recurrence Time

- Consider a M.C. with single recurrent class.
  - recurrent state s
  - $t_i \triangleq$ mean first passage time from i to s

$$t_i = E[\text{\# of steps to reach } s \text{ for the first time, starting from } i]$$

$$= E[\min\{n \geq 0 \mid X_n = s\} \mid X_0 = i\}$$

Observe: Transitions out of s are irrelevant

$\Rightarrow$ Convert s into an absorbing state & make a new M.C.

i.e $P_{ss} = 1$ $P_{sj} = 0$ $\forall j \neq s$

$\Rightarrow$ all other states are transient

$\Rightarrow$ Apply (*) to get

$$\begin{cases} t_i = 1 + \sum_{j=1}^{m} P_{ij} t_j & \forall\ i \neq s \\ t_s = 0 \end{cases}$$

linear sys of Eqn, unique soln

How about mean $\overline{\text{recurrence Time}}$?

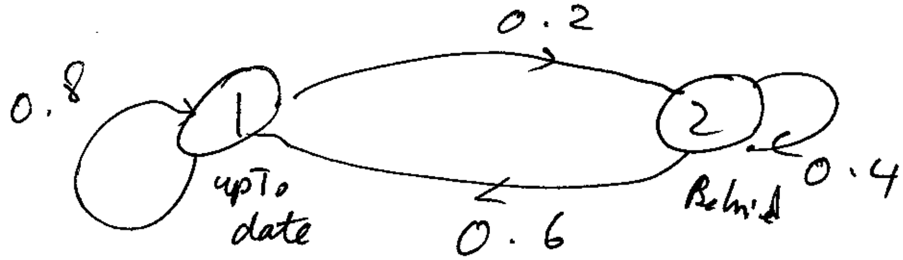
$$t_s^* = E\left\{ \text{number of steps up to first return to } s \text{ starting from } s \right\}$$

$$= E\left[\min\{ n \geq 1 \mid X_n = s \} \middle| X_0 = s \right]$$

$$\overline{t_s^* = 1 + \sum_{j=1}^{m} P_{sj} t_j}$$

<u>Words</u>: Starting from s, time to return to s is 1 plus expected time to reach s from the next state j, with prob P_{sj}

Ex up to date / Behind :



Let $s = 1$. What is mean first passage time to state 1 starting from 2 :

Apply $t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j$
$t_s = 0$
$\Big\} \Longrightarrow$

$t_1 = 0$ $t_2 = 1 + p_{21} t_1 + p_{22} t_2 = 1 + 0.4 t_2$

$\Longrightarrow t_2 = 5/3$

How about mean recurrence

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} \; t_j$$

$$t_1^* = 1 + p_{11} t_1 + p_{12} t_2 = 1 + 0 + 0.2 \cdot \frac{5}{3}$$

$$= \frac{4}{3}$$

Statistics

two schools of thought:

1. Bayesian
2. Classical

} both deal with unknown models or variables

Bayesian: unknowns modeled as random variable with known distributions

Classical: Deterministic unknowns.

Model vs. Variable Inference

Model Inference: Construct a model for a real phenomenon using data

e.g. do planets follow elliptical trajectories

Variable Inference: Use related noisy info to estimate value of unknown variable.

e.g. what is my current position given a few GPS reading.

Ex

binary messages $s_i \in \{0,1\}$

Xmiter sends s_i

Receiver gets $X_i = a\,s_i + W_i \qquad i = 1, 2, \ldots, n$

W_i = zero mean normal R.V. models channel noise.

a = scalar models channel attenuation

Model Inference : a unknown.

sent pilot signal : sequence of message $s_1, \dots s_n$ known to both Xmitter & Rx. $\Rightarrow$

Receiver observes $X_1, \dots X_n$ to estimate a.

Variable Inference : a assumed to be known.

Rx observes $X_1 \dots X_n$ and infers $s_1, \dots s_n$.

Classifying Stat Inference Problems

(1) Binary Hypothesis Testing:

2 hypothesis.

use data to decide which is true.

e.g observe X_i decide whether $s_i = 0$ or 1.

(2) m-ary hypothesis Testing:

. m competiting hypothesis

How to judge performance of Hypothesis Testing?

Prob of erroneous decision.

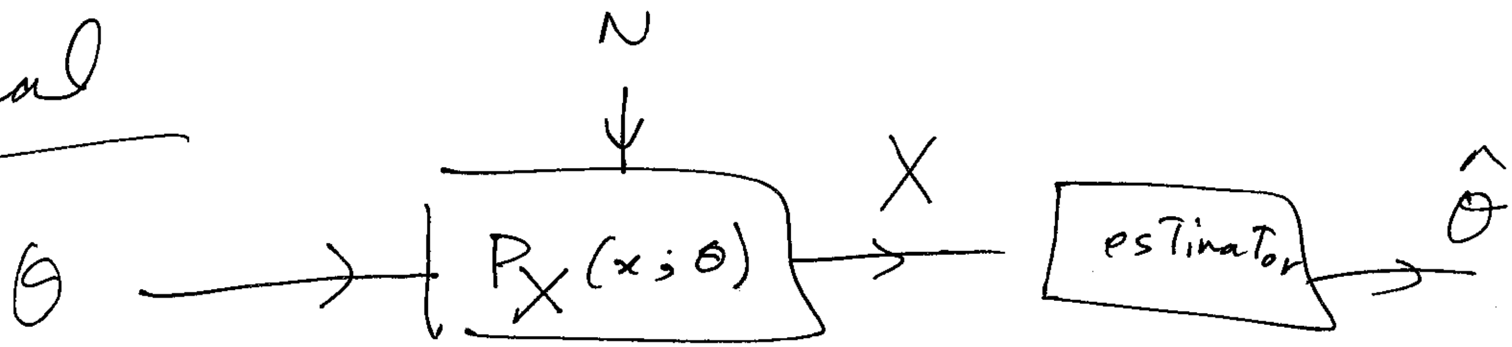
need to minimize this

(3) **Estimation**: model is fully specified except for parameter(s) θ

e.g. estimate a

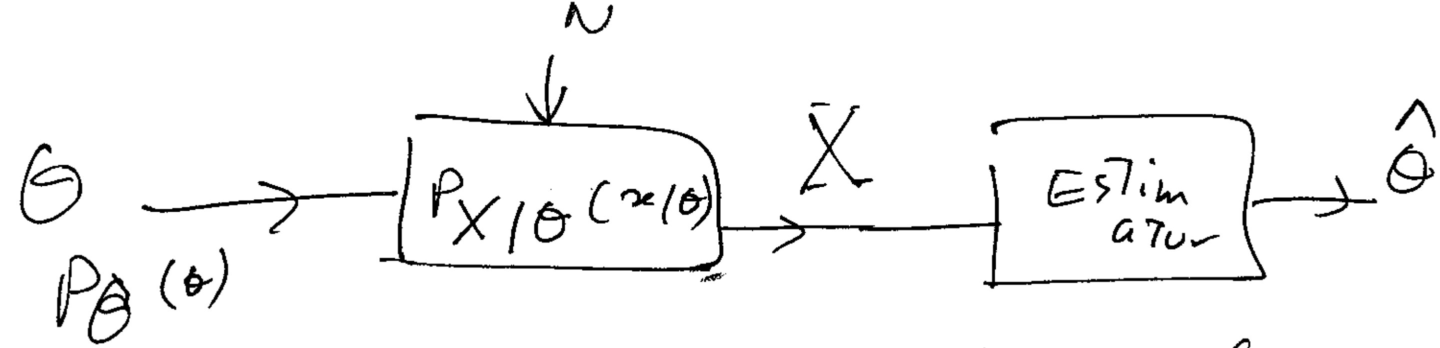
Bayesian and Classical Inference

Classical



θ unknown parameter not a random variable.

Bayesian



- start with prior distribution $P_{\Theta}(\theta)$ for unknown random variable θ
- ~~Have~~ Assume a model $P_{X|\Theta}$ of the observation vector X
- Observe value x of X.
- Form Posterior distribution of Θ using Bayes rule.

$$P_{\Theta|X}(\theta \mid X = x)$$

- Use This to answer questions:
  - estimate Θ
  - estimate probability questions related to θ

Bayes rule

- Assume θ discrete X discrete

$$P_{\Theta|X}(\theta, x) = \frac{P_\Theta(\theta)\, P_{X|\Theta}(x|\theta)}{\sum_{\theta'} P_\Theta(\theta')\, P_{X|\Theta}(x|\theta')}$$

EX. Coin with unknown parameter θ

- observe X heads in n tosses
- Assume θ is uniform (prior)
- Find $f_{\Theta|X}(\theta|x)$

observation discrete, unknown continuous R.V.

$$\text{Bayes Rule} = \frac{f_\Theta(\theta)\, P_{X|\Theta}(x|\theta)}{\int f_\Theta(\theta)\, P_{X|\Theta}(x|\theta)\, d\theta}$$

$P_X(x) = \int f_\Theta(\theta)\, P_{X|\Theta}(x|\theta)\, d\theta$

What to do with the output of Bayesian Inference

pmf $\quad p_{\Theta|X}(\cdot|x)$ or

pdf $\quad f_{\Theta|X}(\cdot|x)$

① Maximum a posterior probability (MAP)

Find θ that maximizes these.

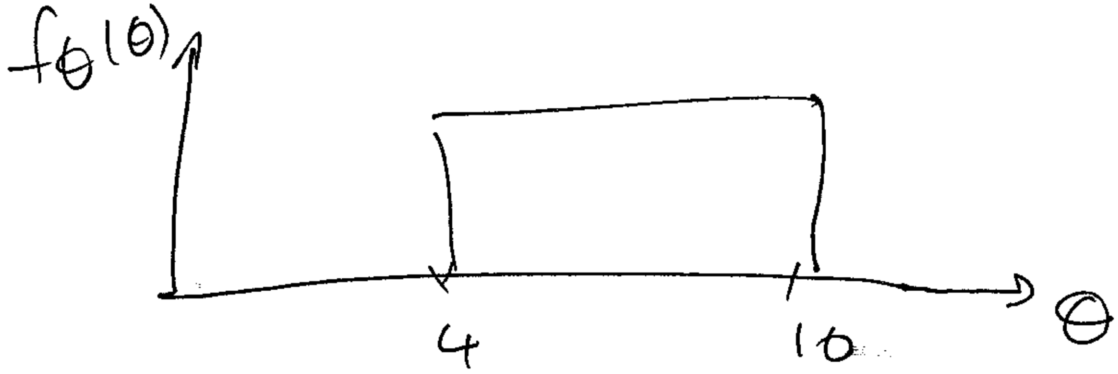
Discrete $\quad p_{\Theta|X}(\theta^*, x) = \max_{\theta} p_{X|\Theta}(\theta|x)$

Continuous $\quad f_{\Theta|X}(\theta^*, x) = \max_{\theta} f_{\Theta|X}(\theta|x)$

② Conditional Expectation:

$$E(\Theta / X = y) = \int \theta f_{\Theta|X}(\theta / y)\, d\theta$$

What if there is no observation at all?



- How to estimate θ? Least mean squares.

- Find c to minimize $E[(\Theta - c)^2]$.

- optimal ~~answer~~ estimate: $c = E(\Theta)$

- How much error in my estimate?

$$E[(\Theta - E(\Theta))^2] = \mathrm{Var}(\Theta)$$