Ex of Total Prob Theorem

- not allowed to exceed 2 rolls.

Ex. Roll a fair 4 sided die.
- If I get 1 or 2, roll again.
- Otherwise stop.

Compute $P(\text{sum of all the rolls is at least 4})$

$P(A_i) = \frac{1}{4}$

- If $A_1$,
  - Either 3, or 4 in roll 2 $\Rightarrow P(B/A_1) = \frac{3}{4}$
  - If $A_2$,
    - (2nd roll 2, 3, 6)
  - If $A_3$,
  - If $A_4$

$P(B/A_2) = \frac{3}{4}$

$P(B/A_3) = 0$

$P(B/A_4) = 1$

$p(b) = p(a_1)P(B/A_1) + p(a_2)P(B/A_2) + p(a_3)P(B/A_3) + p(a_4)P(B/A_4)$

$= \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = 9/6 = 1$
Bayes' Rule

\[ A_1 = \text{malignant} \quad A_2 = \text{non-malignant} \quad A_3 = \text{other} \]

\[ P(B | A_i) \]

\[ P(AB) = P(A)P(B | A) = P(B)P(A | B) \]

\[ \Rightarrow P(A | B) = \frac{P(A)P(B | A)}{P(B)} \]

Bayes' Rule
\[ P(A;1|B) = \frac{P(A_i;1|B)}{P(B)} \]

\[ P(A;1|B) = P(A_i;1|B) = P(A_i;1|B) = \frac{P(A_i)P(B|A_i)}{P(B)} + \frac{P(A_0)P(B|A_0)}{P(B)} + \ldots + \frac{P(A_n)P(B|A_n)}{P(B)} \]

A_i: cause
B = effect

A_i: A_1, A_2, \ldots

An disjoint event. That form a partition of the sample space.
Ex Radar:

\[ A = \{ \text{an aircraft present} \} \]
\[ B = \{ \text{radar says "yes"} \} \]

\[ P(A) = 0.05 \]
\[ P(B/A) = 0.99 \]
\[ P(B/A^c) = 0.1 \]

Compute \( P(\text{aircraft present} \mid \text{radar says "yes"}) \)

\[ P(A/B) = \frac{P(A) \cdot P(B/A)}{P(A) \cdot P(B/A) + P(A^c) \cdot P(B/A^c)} \]

\[ = \frac{0.05 \times 0.99}{0.05 \times 0.99 + 0.95 \times 0.1} \]

\[ P(A/B) = 0.3426 \]

\[ A = \text{true} \]
\[ B = \text{effect} \]
Ex. Chem.

\[ P(A_1) = \frac{1}{2} \]

\[ P(A_2) = \frac{1}{4} \]

\[ P(A_3) = \frac{1}{4} \]

B = winning.

\[ P(B/A_1) = 0.3 \]

\[ P(B/A_2) = 0.4 \]

\[ P(B/A_3) = \frac{1}{2} \]

Assume you win.

\[ P(A_1/B) \]

\[ P(A_1/B) = \frac{P(A_1) \cdot P(B/A_1)}{P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) + P(A_3) \cdot P(B/A_3)} \]

\[ = \frac{\frac{1}{2} \times 0.3}{0.5 \times 0.3 + \frac{1}{4} \times 0.4 + \frac{1}{4} \times 0.3} \]

\[ P(A_1(B)) = 0.4 \]

\[ A = \text{case} \quad B = \text{effect} \]
A, B independence if knowing B happened.

$$P(\frac{A}{B}) = P(A)$$

$$P(B) \neq 0$$

$$P(A/B) = \frac{P(AB)}{P(B)}$$

$$P(AB) = P(A)P(B)$$
Example 2: Successive rolls of a 4-sided die.
- All outcomes equally likely. \( P = \frac{1}{16} \)

(a) \( A_i = \{ \text{first roll is } i \} \)
\( B_j = \{ \text{2nd roll is } j \} \)

\( A_i, B_j \) independent.

**Intuitively:** \( \frac{1}{4} \).

**Formally:**
\[
P(A_i B_j) = \frac{1}{16} \quad \frac{p(A_i)}{p(B_j)} = \frac{1}{4} \times \frac{1}{4}
\]

(b) \( A = \{ \text{1st roll is 1} \} \)
\( B = \{ \text{sum of 2 rolls is 8} \} \)

**Intuitively not independent:**
\[
P(A \mid B) = 0 \quad \Rightarrow \text{not indep.}
\]
\[
P(A) = \frac{1}{4}
\]
Conclusion if event $A$, $B$ are disjoint and both have non-zero prob, then they can NOT be independent.

$A = \{ \text{st roll is 1} \}$  
$B = \{ \text{sum is 5} \}$  

$A, B \text{ independent}$?

$P(A) = \frac{1}{4}$  
$P(B) = \frac{4}{16} = \frac{1}{4}$  

$P(AB) = \frac{1}{16}$  
$P(AB) = P(A) \cdot P(B)$  
$\frac{1}{16} = \frac{1}{4} \cdot \frac{1}{4}$
Ex

\[ A = \{ \text{max of 2 rolls is 2} \} \]
\[ B = \{ \text{min of 2 rolls is 2} \} \]

\[ P(AB) = \frac{1}{16} \]

\[ AB : \{ (2,2) \} \]

\[ A : \{ (1,2), (2,1), (2,2) \} \]

\[ P(A) = \frac{3}{16} \]

\[ B : \{ (2,2), (2,3), (2,4), (3,2), (4,2) \} \]

\[ P(B) = \frac{5}{16} \]

\[ P(AB) = P(A)P(B) \text{ not indp.} \]
Conditional Independence

Two events A, B conditionally independent given event C

\[ P(AB|C) = P(A|C)P(B|C) \]

**Note:** If \( P(A)P(B) = P(AB) \) implies the other.

\[ P(AB|C) = \frac{P(ABC)}{P(C)} \]

\[ P(ABC) = P(C)P(B|C)P(A|BC) \]

\[ P(AB|C) = \frac{P(C)P(B|C)P(A|BC)}{P(C)} \]

\[ = P(B|C)P(A|BC) \]

\[ P(A|C) = P(LA|BC) \]

Given C, chance of A happening is not affected by whether or not B happened.
Ex 2: indp. fair coin tosses

\[ H_1 = \{1\text{st head}\} \]
\[ H_2 = \{2\text{nd head}\} \]
\[ D = \{2\text{ tosses have different results}\} \]

\[ H_1, H_2 \text{ indp.} \]

How about if we condition upon \( D \).

\[ P(H_1 / D) = \frac{1}{2} \]
\[ P(H_2 / D) = \frac{1}{2} \]

\[ P(H_1, H_2 / D) \neq P(H_1 / D) \cdot P(H_2 / D) \]

\[ \Rightarrow \text{ Conditional upon } D, \text{ not indp.} \]
Given B, H, and H2 are independent.

\[ P(H_2) = 0.99 \]
\[ P(H_1 | B) = 0.99 \]
\[ P(H_1 | B^c) = 0.01 \]

\[ P(H) = 0.01 \]

\[ P(\neg H) = 0.99 \]

\[ p(\text{H}_2|H_1) = 0.99 \]

\[ p(\text{H}_1|B, H) = 0.99 \]

\[ p(\text{H}_1|B, \neg H) = 0.01 \]

\[ p(\text{H}_2|B, H) = 0.99 \]

\[ p(\text{H}_2|B, \neg H) = 0.01 \]

\[ p(\text{H}_1|B^c, H) = 0.99 \]

\[ p(\text{H}_1|B^c, \neg H) = 0.01 \]

\[ p(\text{H}_2|B^c, H) = 0.99 \]

\[ p(\text{H}_2|B^c, \neg H) = 0.01 \]

\[ R \rightleftharpoons \text{blue dancer} \]

\[ \text{Red} \]

\[ \text{blue} \]

\[ \text{flip 2 heads to win} \]

\[ \text{Choose at random H2} \]

\[ \text{by 2 coin flips. Blue wins}. \]
0. H₁ and H₂ unconditionally.

$$P(H₁) = P(B) P(H₁ \mid B) + P(B^c) P(H₁ \mid B^c)$$

$$= \frac{1}{2} 0.99 + \frac{1}{2} 0.01 = \frac{1}{5}$$

$$P(H₂) = \frac{1}{2}$$

$$P(H₁, H₂) = P(B) P(H₁H₂ \mid B) + P(B^c) P(H₁H₂ \mid B^c)$$

$$= \frac{1}{2} 0.99 \times 0.99 + \frac{1}{2} 0.01 \times 0.01$$

$$= \frac{1}{2}$$

= \frac{1}{2} \text{ not ind.}$$

= \frac{1}{2} \text{ not ind.}$$

= \frac{1}{2} \text{ not ind.}$$

= \frac{1}{2} \text{ not ind.}$$

= \frac{1}{2} \text{ not ind.}$$

= \frac{1}{2} \text{ not ind.}$$

= \frac{1}{2} \text{ not ind.}$$