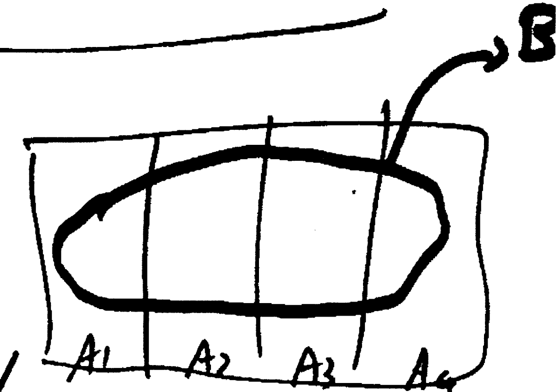
Ex of Total Prob Thm

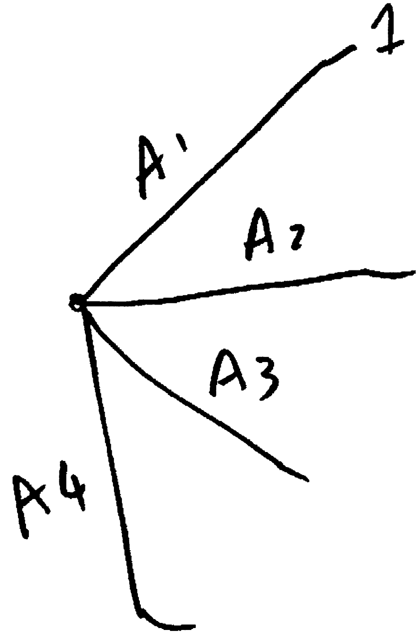
- not allowed to exceed 2 rolls.

Ex - Roll a fair 4 sided die.
- If I get 1 or 2, roll again.
  otherwise stop.

Q: Compute P($\underbrace{\text{sum of all the rolls is at least 4}}_{B}$)

$P(A_i) = 1/4$





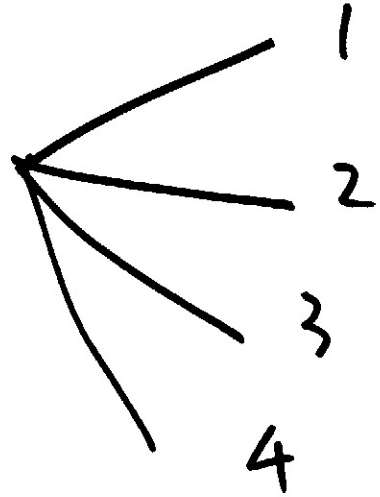
- If A_1, (either 3, or 4 in roll 2) $\Rightarrow P(B|A_1) = 2/4$
- If A_2, (2nd roll 2, 3, 4) $P(B|A_2) = 3/4$
- If A_3, $P(B|A_3) = 0$
- If A_4, $P(B|A_4) = 1$

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)$$

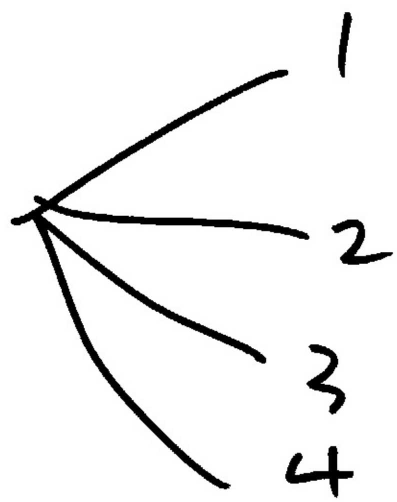
$$= \frac{1}{4}\frac{2}{4} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1$$

$$= 9/16$$

- roll 1 = 1

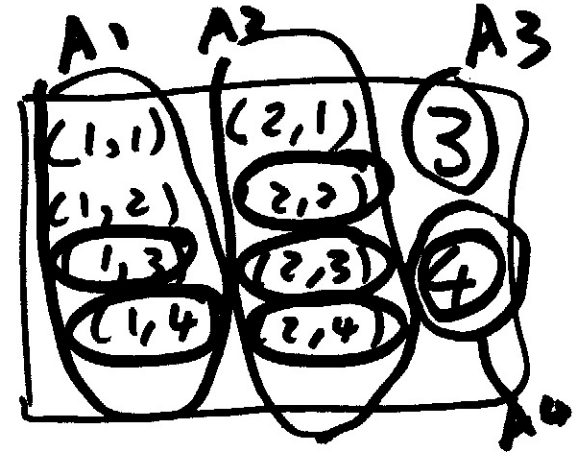


roll 1 = 2



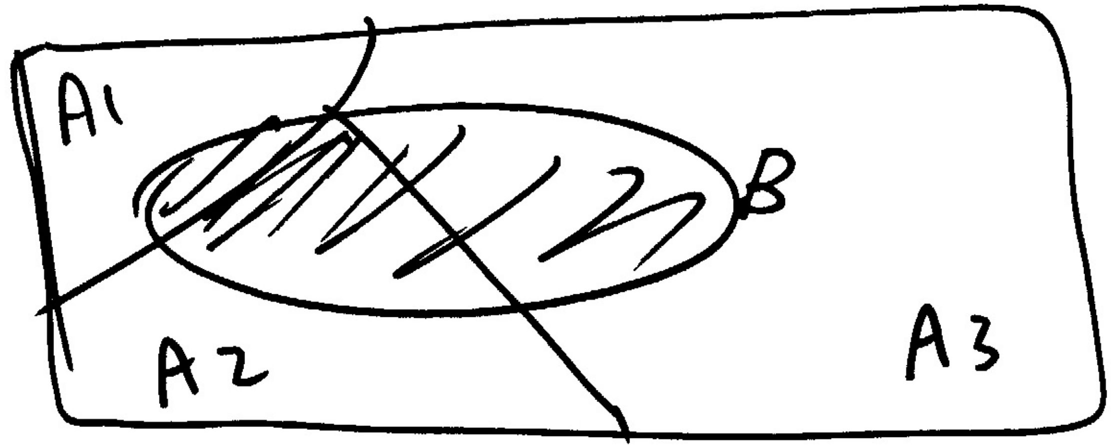
roll 1 = 3 → 3

roll 1 < 4



red = B

Bayes Rule



A_1 = malignant $\quad A_2$ = non malignant $\quad A_3$ = others.

$$P(B \mid A_i)$$
$$\downarrow$$
$$P(A_i \mid B)$$

$$P(AB) = P(A)P(B|A) = P(B)\,P(A|B)$$

$$\Longrightarrow P(A|B) = \frac{P(A)\,P(B|A)}{P(B)} \leftarrow \text{Bayes Rule}$$

$A_1, A_2, \ldots A_n$ disjoint events. That form a Partition of the sample space

$$P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{P(B)}$$

$$P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{P(A_1) P(B \mid A_1) + P(A_2) P(B \mid A_2) + \cdots \cdots P(A_n) P(B \mid A_n)}$$

A_i = cause $\quad B$ = effect.

<u>Ex</u> Radar. $A = \{\text{an aircraft present}\}$

$B = \{\text{radar says "yes"}\}$

$P(A) = 0.05$ $\quad P(B|A) = 0.99$

$P(B|A^c) = 0.1$

Compute $P(\text{aircraft present} \mid \text{radar registers})$

$= P(A \mid B)$

$$= \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(A^c)\,P(B|A^c)}$$

$$= \frac{0.05 \times 0.99}{0.05 \times 0.99 + 0.95 \times 0.1}$$

$P(A|B) = 0.3426$

$A =$ cause

$B =$ effect.

Ex chess.

$P(A_1) = 1/2$ $\quad$ $P(A_2) = 1/4$ $\quad$ $P(A_3) = 1/4$

B = winning.

$P(B|A_1) = 0.3$ $\quad$ $P(B|A_2) = 0.4$ $\quad$ $P(B|A_3) = 1/2$

Assume you win. what are chances you played against Type 1

$$P(A_1|B)$$

$$P(A_1|B) = \frac{P(A_1)\,P(B|A_1)}{P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + P(A_3)\,P(B|A_3)}$$

$$= \frac{1/2 \times 0.3}{0.5 \times 0.3 + \frac{1}{4} \cdot 0.4 + \frac{1}{4} \cdot 0.3}$$

$$P(A_1|B) = 0.4$$

A = cause $\quad$ B = effect

Independence

A, B indep if knowing B happened doesn't change chances of A happening.

<u>Math</u> $\boxed{P(A/B) = P(A)}$

$P(B) \neq 0$

$$P(A/B) = \frac{P(AB)}{P(B)} = P(A) \Longrightarrow$$

$$\boxed{P(AB) = P(A)\,P(B)}$$

__Ex__ 2 successive rolls of 4 sided die.
- all outcomes equally likely. $P = 1/16$

__Q__ $A_i = \{\text{first roll is } i\}$
$B_j = \{\text{2nd " " } j\}$

A_i, B_j independent.

__Intuitively__ yes.

__Formally__:

$P(A_i B_j) = 1/16 \qquad P(A_i) = 1/4$
$\qquad\qquad\qquad\qquad\quad P(B_j) = 1/4$

$P(A_i B_j) = P(A_i)\, P(B_j) = \frac{1}{4} \times \frac{1}{4}$

__Q__ $A = \{\text{1st roll is } 1\} \qquad B = \{\text{sum of 2 rolls is } 8\}$

Intuitively not indep't.

$$\left.\begin{aligned} &P(A \mid B) = 0 \\ &P(A) = 1/4 \end{aligned}\right\} \Rightarrow \text{not indp.}$$

Conclusion if event A, B are disjoint and both have nonzero prob., then they can **NOT** be independent.

Q $A = \{1\text{st roll is } 1\}$ $B = \{\text{sum is } 5\}$

A, B indep?

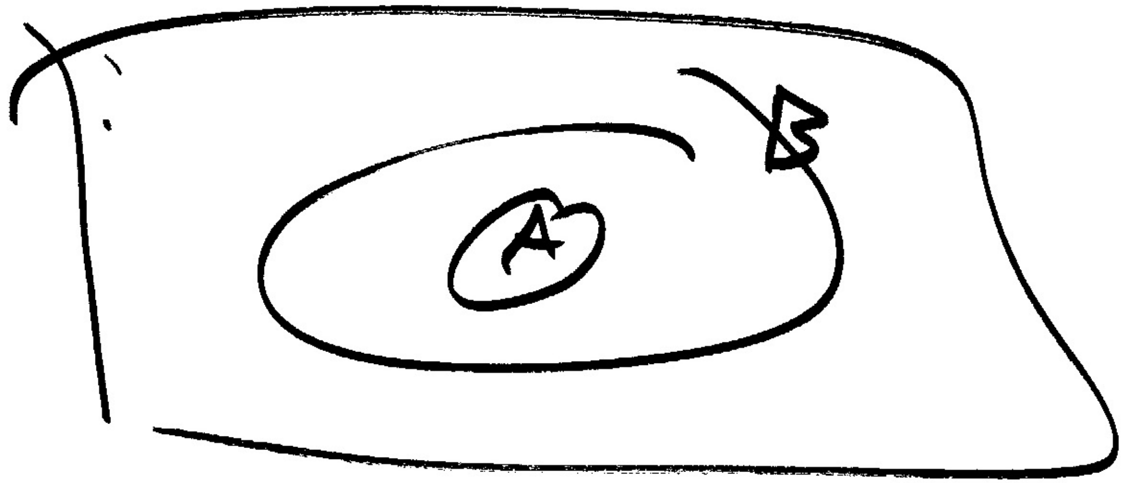
$P(A) = \frac{1}{4}$

$B: \begin{pmatrix} 1,4 \\ 2,3 \\ 3,2 \\ 4,1 \end{pmatrix}$ $P(B) = \frac{4}{16} = \frac{1}{4}$

$P(AB) = \frac{1}{16}$

$P(AB) = P(A)\,P(B)$

$\frac{1}{16} = \frac{1}{4} \cdot \frac{1}{4}$


A
B

<u>Ex</u> $A = \{$ max of 2 rolls is 2 $\}$

$B = \{$ min of 2 rolls is 2 $\}$

$$P(AB) = \frac{1}{16}$$

$AB : \{(2,2)\}$

$A : \{(1,2), (2,1), (2,2)\}$ $\quad P(A) = \frac{3}{16}$

$B : \{(2,2)\ (2,3)\ (2,4), (3,2)\ (4,2)\}$ $\quad P(B) = 5/16$

$P(AB) \neq P(A)P(B) \to$ not indp.

Conditional Indp.

Two event A, B conditionally indp given event C, if:

$$P(AB \mid C) = P(A \mid C)\, P(B \mid C) \quad (*)$$

Note: If $P(A)P(B) = P(AB)$ → neither one implies the other

$$P(AB \mid C) = \frac{P(ABC)}{P(C)}$$

$$P(ABC) = P(C)\,P(B \mid C)\,P(A \mid BC)$$

$\Longrightarrow$

$$P(AB \mid C) = \frac{P(C)\,P(B \mid C)\,P(A \mid BC)}{P(C)}$$

$$= P(B \mid C)\,P(A \mid BC) \quad (*)$$

$(*) \Longrightarrow \boxed{P(A \mid C) = P(A \mid BC)}$

Given C, chances of A happening is not effected by whether or not B happened.

Ex 2 ~~i~~ indp. fair coin Tosses

$H_1 = \{ \text{1st head} \}$ $\qquad$ $D = \{ \text{2 Tosses have diff results} \}$.

$H_2 = \{ \text{2nd head} \}$

$$\boxed{\{H_1 T_2,\ T_1 H_2\}} \to D$$

H_1 H_2 indp.

How about if we condition upon D.

$P(H_1 | D) = \frac{1}{2}$ $\qquad$ $P(H_1 H_2 | D) = 0$

$P(H_2 | D) = \frac{1}{2}$

$P(H_1 H_2 | D) \neq P(H_1 | D) \cdot P(H_2 | D)$

$\Rightarrow$ conditional upon D, not indp.

Ex 2 coins: blue / red.

choose at random 1/2

flip 2 indp tosses.

blue.	$P(h) = 0.99$	$p(t) = 0.01$
red	$P(h) = 0.01$	$P(t) = 0.99.$

$B =$ blue chosen

$H_i =$ ith toss is head.

Given B, H_1 and H_2 are independent

$$\left.\begin{aligned} P(H_1 H_2 \mid B) &= 0.99 \times 0.99. \\ P(H_1 \mid B) &= 0.99 \\ P(H_2 \mid B) &= 0.99 \end{aligned}\right\} \Longrightarrow$$

(1) H_1 and H_2 unconditionally indp.

No

$$P(H_1) = P(B)\,P(H_1 \mid B) + P(B^c)\,P(H_1 \mid B^c)$$

$$= \frac{1}{2}\,0.99 + \frac{1}{2}\cdot 0.01 = \frac{1}{2}$$

$$P(H_2) = \frac{1}{2}$$

$$P(H_1 H_2) = P(B)\,P(H_1 H_2 \mid B) + P(B^c)\,P(H_1 H_2 \mid B^c)$$

$$= \frac{1}{2}\,0.99 \times 0.99 + \frac{1}{2}\,0.01 \times 0.01 \approx \frac{1}{2}$$

Not indp.